Dihedral-like constructions of automorphic loops

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Review facts about dihedral groups and automorphic loops.

Definition(Dihedral groups)

The **dihedral group** of degree n and order $2n$, denoted as Dih_n is the group generated by two elements a and b with multiplication determined by $b^n = a^2 = 1$ and $a \cdot b = b^{n-1} \cdot a$.

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Definitions

For a loop Q define	
Left and right translations of y by x	$L_x(y) = xy$ and $R_x(y) = yx$
Multiplication group of Q	$Mlt(Q) =< L_x, R_x : x \in Q > .$
Inner mapping group of Q	$Inn(Q) = (Mlt(Q))_1.$
Automorphism group of Q	$Aut(Q)$ = the automorphism group of G.

Automorphic loops

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A loop is automorphic (or an **A-loop**) if every inner mapping is an automorphism, that is, if $\mathrm{Inn}(Q) \leq Aut(Q)$.

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Proposition (Bruck and Paige)

A loop Q is an automorphic loop if and only if , for all $x, y, u, v \in Q$

$$(uv)R_{x,y} = uR_{x,y} \cdot vR_{x,y}, \qquad (A_r)$$

$$(uv)L_{x,y} = uL_{x,y} \cdot vL_{x,y}, \qquad (A_l)$$

$$(uv)T_x = uT_x \cdot vT_x. \qquad (A_m)$$

Remark

To check that a particular loop is automorphic, it is not necessary to verify all of the conditions $(A_r), (A_\ell)$ and (A_m).

Automorphic loops

Proposition(Johnson, Kinyon, Nagy and Vojtěchovský.)

Let Q be a loop satisfying (A_m) and (A_l). Then Q is automorphic.

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Definitions

Commutator	$xy = (yx) \cdot [x, y]$, for $x, y \in Q$.
Associator	$(xy)z = x(yz) \cdot [x, y, z]$, for $x, y, z \in Q$.
Commutant	$C(Q) = \{x \in Q : xy = yx \text{ for every } y \in Q\}$

Definitions

For a loop Q define

Left nucleus of Q	$N_\lambda = \{a \in Q \mid ax \cdot y = a \cdot xy, \forall x, y \in Q\}$.
Right nucleus of Q	$N_\rho = \{a \in Q \mid xy \cdot a = x \cdot ya, \forall x, y \in Q\}$.
Middle nucleus of Q	$N_\mu = \{a \in Q \mid xa \cdot y = x \cdot ay, \forall x, y \in Q\}$.
Nucleus of Q	$N(Q) = N_\lambda(Q) \cap N_\rho(Q) \cap N_\mu(Q)$.

Generalization

Generalized dihedral loop

For an integer $m \geq 1$, an abelian group $(G, +)$ and an automorphism α of G, define $Dih(m, G, \alpha)$ on $\mathbb{Z}_m \times G$ by

$$(i, u) \cdot (j, v) = (i + j, (s_j u + v)\alpha^{ij}), \qquad (1)$$

where $s_i = (-1)^{i \mod m}$, and we interpreted α^{ij} as

- Interpret α^{ij} as $\alpha^{ij \mod m}$, is called the dihedral reducing modulo m, and it is denoted by $Dih_{red}(m, G, \alpha)$. In these calculations we no more have $\alpha^i \alpha^j = \alpha^{i+j}$.
- interpret α^{ij} as ordinary integral exponent, is called the dihedral not reducing modulo m, and it is denoted by $Dih(m, G, \alpha)$. We demand that $i, j \in \{0, \ldots, m-1\}$ and we have $\alpha^i \alpha^j = \alpha^{i+j}$, so the multiplication formula is unambiguous.

Generalized dihedral loop

Remark

- Note that when $m = 2$ the two interpretations coincide. This is because $\alpha^{ij} = \alpha^{ij \mod m}$ for every $i, j \in \{0, 1\}$. But for $m > 2$ the two interpretations do not coincide in general. For instance, if $m = 3$ and $|\alpha| > 3$ then $\alpha^2\alpha^2 = \alpha^4 \neq \alpha = \alpha^{2 \cdot 2 \mod 3}$.
- $Dih(m, G, \alpha)$ is not necessarily isomorphic to $Dih_{red}(m, G, \alpha)$. There are examples of order 20. It turns out that $Dih(m, G, \alpha)$ is an automorphic loop if and only if $Dih_{red}(m, G, \alpha)$ is an automorphic loop, in which case $Dih(m, G, \alpha) \cong Dih_{red}(m, G, \alpha)$.In such case the two constructions coincide. So it suffices to work with only one of the constructions.

Main result

Question

Which choices of parameters m, G, α make the loop $Dih(m, G, \alpha)$ an automorphic loop, particularly a nonassociative automorphic loop?.

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Main Theorem

Let $Q = Dih(m, G, \alpha)$,

(i) If $m = 2$ then Q is automorphic.

(ii) If $m > 2$ is even then Q is automorphic iff $\alpha^2 = \mathrm{id}$.

(iii) If $m > 2$ is odd then Q is automorphic iff $\alpha = \mathrm{id}$ and $2G = 0$, in which case Q is a group.

Middle Nucleus and Commutant

Propositon

Let $Q = Dih(m, G, \alpha)$. If m is even and $\alpha^2 = id$ then $N_\mu =< 2 > \times G$. In particular when $m = 2$ then $N_\mu = \{0\} \times G \cong G$.

Middle Nucleus and Commutant

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Corollary

Let $Q = Dih(m, G, \alpha)$, m is even and $\alpha^2 = id$. Then $N_\mu(Q)$ is an abelian group.

Corollary

Let $Q = Dih(m, G, \alpha)$, m is even and $\alpha^2 = id$, with middle nucleus $N_\mu(Q)$.Then $[Q : N_\mu(Q)] = 2$.

Middle Nucleus and Commutant

Lemma

Let $Q = Dih(m, G, \alpha), m$ is even and $\alpha^2 = id$.

(i) If $\exp G \leq 2$ then $C(Q) = Q$.

(ii) If $\exp G > 2$ then $(i, u) \in C(Q)$ iff i is even and $|u| \leq 2$.

(iii) $C(Q)$ is a normal subloop of Q.

Left Commutators and Associators

Left commutator

- The left commutator $[(i,u),(j,v)]$ is

$$(0, \Big((s_j - 1)u + (1 - s_i)v\Big)\alpha^{ij})$$

- $[(i,u),(j,v)] \in 0 \times 2G$.

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lemma

Let $Q = Dih(m, G, \alpha)$, m is even and $\alpha^2 = id$. Then $0 \times 2G$ is normal subloop of Q.

Left Commutators and Associators

Left associator

- Left associator $[(i,u),(j,v),(k,w)]$ is

$$(0,(s_{j+k}u(1-\alpha^{-jk})\alpha^{ij}+w(1-\alpha^{ij}))\alpha^{(i+j)k})$$

  for all $i,j,k \in \mathbb{Z}_m, u,w \in G$.
- $[(i,u),(j,v),(k,w)] \in 0 \times (1-\alpha)G$.

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Lemma

Let $Q=Dih(m,G,\alpha)$, m is even and $\alpha^2=id$. Then $0\times(1-\alpha)G$ is normal subloop of Q.

lemma

Let $Q=Dih(m,G,\alpha)$, m is even and $\alpha^2=id$. Then $A(Q)=0\times(1-\alpha)G$.

Commutators and Associators in Dihedral A-loop

lemma

Let $Q = Dih(m, G, \alpha)$, m is even and $\alpha^2 = id$. Then $Q' = (1 - \alpha)G \cup 2G$.

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lemma

Let $Q = Dih(m, G, \alpha)$, m is even and $\alpha^2 = id$. $A(Q)$ and Q' are normal subloop of $N_\mu(Q)$.

Isomorphism

Conjecture

Fixing m, G. Given $\alpha, \beta \in Aut(G), (\forall m > 2$, also assume that $|\alpha|, |\beta| \leq 2)$, then $Dih(m, G, \alpha) \cong Dih(m, G, \beta)$ iff α and β are conjugated (that is there is γ such that $\alpha = \gamma\beta\gamma^{-1}$).

Isomorphism

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proof

We want to show that if α and β are conjugated then $Dih(m, G, \alpha) \cong Dih(m, G, \beta)$. Supposing α and β are conjugated. Then there exist an isomorphism $\gamma : G \longrightarrow G$ such that $\alpha = \gamma\beta\gamma^{-1}$.

Define $\phi : Dih(m, G, \alpha) \longrightarrow Dih(m, G, \beta)$ by $\phi(i, u) = (i, u\gamma)$. It is easily to check that ϕ is a bijection and a homomorphism.

Next let's assume that $Dih(m, G, \alpha) \cong Dih(m, G, \beta)$. We would like to show that α and β are conjugated ...

Thank You For Your Attention!