Towards a Geometric Theory for Left Loops

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- I present some characterization theorems for Cayley graphs for different algebraic structures.
- I introduce the idea of a Geometric Left Loop Theory.

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- $a \in (as)S \quad \forall a \in M, \forall s \in S$

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Let M be a magma, and let $S \subset M$ be a Cayley set. The Cayley graph of M with respect to S is $Cay(M, S) = (V, E)$ where $V = M$ and $E = \{\{x, xs\} : x \in M, s \in S\}$.

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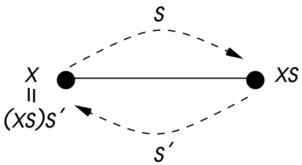
The condition $a \notin aS \quad \forall a \in M$ of the Cayley set implies that there are no loop-edges in the Cayley graph.
The condition $a \in (as)S \quad \forall a \in M, \forall s \in S$ of the Cayley set implies that the Cayley graph is undirected.



Definition

Let M be a magma. A subset $S \subset M$ is called quasi-associative if $(ab)S = a(bS)$ for all $a, b \in M$.

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- *The equation $ax = b$ has a unique solution $x = a \backslash b$.*
- *There exists $e \in L$ such that $ae = a$ for all $a \in L$.*

Definition

A graph $X = (V, E)$ is vertex-transitive *if for every $x, y \in V$ there exists a graph automorphism σ such that $\sigma(x) = y$.*

Theorem

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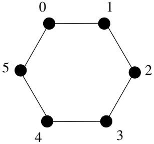
The answer is NO, as the following counter-example shows.

Counterexample

[KB] *Let* $L = \{0,1,2,3,4,5\}$. *Define in* L *the binary operation* $*$ *given by the following table:*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	4	5	3	0
2	2	3	5	4	0	1
3	3	4	0	1	5	2
4	4	5	1	0	2	3
5	5	0	3	2	1	4

Note that L *is a left loop, moreover, is a loop (the identity being 0). One can verify that th set* $S = \{1,5\}$ *is a Cayley set, and that the graph* $Cay(L,S)$ *is the cycle* C_6, *which is vertex transitive.*



Nevertheless, $(2*3)*S = \{5,3\}$ *while* $2*(3*S) = \{0,5\}$, *so* S *is not quasi-associative.*

Characterization of Vertex-transitive Graphs

Theorem

[Mwambené] *Let* $X = (V, E)$ *be a vertex-transitive graph. Then there exists a left loop* L *and a quasi-associative Cayley set* $S \subset L$ *such that* $Cay(L, S) \cong X$*.*

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This theorem together with the previous one, give a characterization of vertex-transitive graphs as Cayley graphs of left loops with respect to quasi-associative Cayley sets.

Mwambené's proof is constructive: starting from a vertex-transitive graph X, he constructs a left loop L and a quasi-associative Cayley set S such that $Cay(L, S)$ is isomorphic to X.

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The construction is the following:

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- We define $S_T := \{\sigma \in T : \{u, \sigma(u)\} \in E\}$.

It turns out that S_T is a quasi-associative Cayley set, and that $Cay(T, S_T) \cong X$.

With Mwambené's method one can reconstruct many left loops (one for each transversal T) and their corresponding quasi-associative Cayley sets S_T such that $Cay(T, S_T) \cong X$.

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A valid question is: Are these all the left loops with a quasi-associative Cayley such that the Cayley graph is isomorphic to the given vertex-transitive graph?
The answer is YES.

Theorem

[KB] *Let L be a left loop and let $S \subset L$ be a quasi-associative Cayley set. Let $X = Cay(L, S)$. Then, there exists a left loop T constructed by Mwambené's method starting from the graph X, which is isomorphic to L. Moreover, there exists an isomorphism $\varphi : L \to T$, such that $\varphi(S) = S_T$.*

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This means that given the graph $Cay(L, S)$, one can reconstruct the left loop L with Mwambené's method, and moreover, the Cayley set constructed by Mwambené's method coincides with the original quasi-associative Cayley set.

Remark

According to the previous theorem, one can construct every left loop **with a quasi-associative Cayley set** *from it's Cayley graph. But the theorem is useless when the Cayley set is not quasi-associative (like in the previous counterexample).*

Mwambené's method can be used to prove some other characterization theorems for Cayley graphs.

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Recall

If X is any set and $\Omega \subset Sym(X)$, we say that Ω acts regularly in X if for every $x, y \in X$ there's a **unique** *$\sigma \in \Omega$ such that $\sigma(x) = y$.*

Theorem

[Gauyacq] *A graph* $X = (V, E)$ *is isomorphic to the Cayley graph of a* **quasi-group** Q *with respect to a quasi-associative Cayley set* S *if and only if* $Aut(X)$ *contains a* **subset** T *that acts regularly on* V.

Theorem

A graph $X = (V, E)$ is isomorphic to the Cayley graph of a **loop** *L with respect to a quasi-associative Cayley set S if and only if $Aut(X)$ contains a* **subset** *T that acts regularly on V* **and** *$Id \in T$.*

Theorem

[Sabidoussi] *A graph* $X = (V, E)$ *is isomorphic to the Cayley graph of a* **group** G *with respect to a Cayley set* S *if and only if* $Aut(X)$ *contains a* **subgroup** T *that acts regularly on* V*.*

In the next section I will try to introduce a Geometric Left Loop Theory in analogy to Geometric Group Theory.

Quasi-isometry

Definition

Let (X, d) and (X', d') be metric spaces, and let $f : X \to X'$. Let $\lambda > 0, k \geq 0$. f is a (λ, k)-quasi-isometry if for every $x, y \in X$

$$\frac{1}{\lambda} d(x, y) - k \leq d'(f(x), f(y)) \leq \lambda d(x, y) + k. \qquad (1)$$

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Remarks

- A quasi-isometry is not necessarily inyective nor continous.
- If there's a quasi-isometry from X to X', there's not necessarily one from X' to X.

Almost Surjective

Definition

Let (X, d) and (X', d') be metric spaces, and let $f : X \to X'$. It is said that f is almost surjective if there exists $\delta \geq 0$ such that

$$\forall x' \in X' \, \exists x \in X : \quad d'(f(x), x') \leq \delta \tag{2}$$

Quasi-isometric Spaces

Proposition

If there exists an **almost surjective** (λ, k)*-quasi-isometry from a metric space* X *to a metric space* X'*, then there's an almost surjective* (λ', k')*-quasi-isometry from* X' *to* X*.*

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Being quasi-isometric is an equivalence relation.

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Let G be a group and let $S \subset G$ be a Cayley set. Then $Cay(G, S)$ is connected if and only if $G =< S >$.

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Theorem

Let G be a finitely generated group, and let S and S' be finite Cayley sets that generate G. Then the graphs $Cay(G, S)$ and $Cay(G, S')$ are quasi-isometric.

In general, if we have a magma M and a Cayley set S, the connected component of a vertex a, are the elements of the form

$$x = (\dots(((as_1)s_2)s_3)\dots)s_k, \tag{3}$$

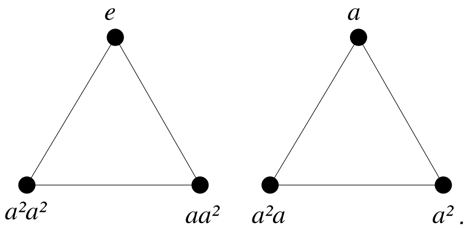
where $s_i \in S$. Therefore the previous results are false in the more general case. In fact we have the following counterexample:

Counterexample

Let L be the left loop given by the following table:

$*$	e	a	a^2	a^2a	aa^2	a^2a^2
e	e	aa^2	a	a^2a^2	a^2	a^2a
a	a	a^2	aa^2	a^2a	e	a^2a^2
a^2	a^2	a^2a	a^2a^2	a	aa^2	e
aa^2	aa^2	a^2a^2	a^2	e	a^2a	a
a^2a	a^2a	a	e	a^2	a^2a^2	aa^2
a^2a^2	a^2a^2	e	a^2a	aa^2	a	a^2

One can verify that $S = \{a, a^2a\}$ *is a Cayley set, and clearly, it generates L (everything is in terms of a). Nevertheless, the Cayley graph is the following:*



Next we want to prove that these results are true if we replace the word “group” by “left loop” **and** we ask the Cayley set S to be **quasi-associative**.

Products and Normed Products

Theorem

[KB] *Let M be a magma and let $S \subset M$ be a quasi-associative set. Then every product of lenght k of elements of S,*

$$x = s_1 s_2 ... s_k \quad s_i \in S \; \forall i = 1, \ldots k \tag{4}$$

with any parenthesis arrangement, can be written also as a left normed product of the same length. That is,

$$x = (\ldots((s'_1 s'_2) s'_3) \ldots) s'_k \quad s'_i \in S \; \forall i = 1, \ldots k. \tag{5}$$

Distance in the Cayley Graph of a Left Loop

Proposition

[KB] *Let L be a left loop and let $S \subset L$ be a quasi-associative Cayley set. Let $x, y \in L$. Then the distance from x to y in the graph $Cay(L, S)$ is the minimal length of a product expressing $x \backslash y$.*

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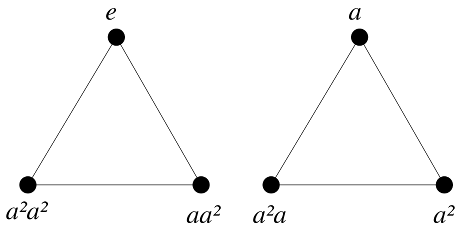
NOTE: This result, which is obvious in the case when L is a group, is not true without the condition of S being quasi-associative.

In the previous counterexample:



If we calculate $a \backslash (aa^2)$ we get a^2 that has length 2. But $d(a, aa^2) = \infty \neq 2$.

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[KB] *Let L be a left loop and let $S \subset L$ be a quasi-associative Cayley set. Then $Cay(L, S)$ is connected if and only if $L =< S >$.*

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Theorem

[KB] *Let L be a left loop and let S and S' be two finite quasi-associative Cayley sets that generate L. Then the Cayley graphs $Cay(L, S)$ and $Cay(L, S')$ are quasi-isometric.*

This last result is the most important, since it says that every geometric property of the Cayley graph, that is a quasi-isometric invariant (i.e. that if it holds for some space, it holds also for every quasi-isometric space to that one), is an intrinsic property of the left loop L and does not depend on the quasi-associative Cayley set S.

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Hyperbolicity

Definition

(informal)

A metric space is called hyperbolic if there exists $\delta > 0$ such that for any triangle ABC, and for every point x in the segment AC, there exists a point y either on the segment AB or in the segment BC, such that the distance between x and y is less than δ.

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Given two quasi-isometric metric spaces X and X', if one of them is hyperbolic, then the other is hyperbolic too (maybe with a different δ). That is, hyperbolicity is a **quasi-isometric invariant***.*

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Definition

[KB] *Let L be a left loop and let $S \subset L$ be a finite quasi-associative Cayley set that generates L. L is called a hyperbolic left loop if the Cayley graph $Cay(L, S)$ is hyperbolic.*

Remark

By a previous theorem, given a hyperbolic vertex-transitive graph, we can use Mwambené's method to obtain **all** *the hyperbolic left loops with the given graph as a Cayley graph.*

Growth

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Let $X = (V, E)$ *be a vertex-transitive graph and let* $u \in V$ *be a fixed vertex. The growth function of* X *is the function* $\gamma : \mathbb{N} \to \mathbb{N}$ *defined as follows:*

$$\gamma(n) = |\{x \in V : \ d(x, u) \leq n\}|. \tag{6}$$

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Note that since X is vertex-transitive, γ does not depend on the choice of u.

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Let $X = (V, E)$ be a vertex-transitive graph and let $u \in V$ be a fixed vertex. The growth function of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$\gamma(n) = |\{x \in V : \ d(x, u) \leq n\}|. \tag{6}$$

Note that since X is vertex-transitive, γ does not depend on the choice of u.

Definition

Let $f, g : \mathbb{N} \to \mathbb{N}$ be two non-decreasing functions. We define an equivalence relation $\sim$ given by:

$$f \sim g \Leftrightarrow \exists C > 0 : \ g(\frac{1}{C}n) \leq f(n) \leq g(Cn). \tag{7}$$

Proposition

If X and X' are two quasi-isometric vertex-transitive graphs with growth functions γ and γ' respectively, then $\gamma \sim \gamma'$.

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Note that if one changes the choice of S, one gets a quasi-isometric Cayley graph, and, by the previous proposition, the rate of growth is the same. That means that the rate of growth does not depend on the choice of S, but only on the left loop L itself.

Thank you!