Extensions of nilpotent Lie algebras of type 2

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(joint work with Daniel de-La-Concepción)

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Context

Levi's theorem (E. E. Levi, 1905)

Any finite dimensional Lie algebra of characteristic zero splits as sum of its solvable radical and a semisimple algebra.

So, given $\mathfrak{g}$ Lie and $\mathfrak{r}$ its solvable radical, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ with $\mathfrak{s}$ s/s. We note that:

- $\mathfrak{s}$ is called Levi subalgebra of $\mathfrak{g}$;
- $\mathfrak{r}$ is an $\mathfrak{s}$-módulo via the adjoint representation, $\mathrm{ad}_{\mathfrak{r}}$; in fact, $\mathrm{ad}_{\mathfrak{r}}\,\mathfrak{s}$ is a subalgebra of $\mathrm{Der}\,\mathfrak{r}$, Lie algebra of derivations of $\mathfrak{r}$.

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A. I.. Malcev, 1944

Any faithful[a][b] Lie algebra with solvable radical $\mathfrak{r}$ is isomorphic to a Lie algebra of the form ${}^c\mathfrak{g} = \mathfrak{s} \oplus_{id} \mathfrak{r}$ where $\mathfrak{s}$ is a semisimple subalgebra of a Levi subalgebra of $\mathrm{Der}\,\mathfrak{r}$.

Moreover, given $\mathfrak{s}_1$ and $\mathfrak{s}_2$ semisimple subalgebras of $\mathfrak{s}_0$, the algebras $\mathfrak{s}_1 \oplus_{id} \mathfrak{r}$ and $\mathfrak{s}_2 \oplus_{id} \mathfrak{r}$ are isomorphic if and only if $\mathfrak{s}_2 = A\mathfrak{s}_1 A^{-1}$, where A is an authomorphism of $\mathfrak{r}$, named $\mathrm{Aut}\,\mathfrak{r}$.

[a] without semisimple ideals.

[b] Any Lie algebra $\mathfrak{a}$ decomposes as a direct sum of ideals $\mathfrak{a} = \mathfrak{s}' \oplus \mathfrak{g}$, with $\mathfrak{s}'$ semisimple and $\mathfrak{g}$ faithful.

[c] $\mathfrak{s}$ and $\mathfrak{r}$ viewed as subalgebras of $\mathfrak{g}$, and $[s, a] = s(a)$ for $s \in \mathfrak{s}$ and $a \in \mathfrak{r}$.

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Problem

Classification of Lie algebras of a given radical

- Which radical?

  Following Malcev's 1944, solvable radical $\mathfrak{r}$.

  Following Malcev's 1945, (...) *solvable Lie algebras can be classify through nilpotent ones* (...)

  Following Sato's 1971, (...) *nilpotent Lie algebras are quotients of free nilpotents* ones(...)

  Start point : NILRADICAL FREE NILPOTENT

- Which tools?

  Derivations [to construct ...] and automorphisms of the (((free-)nil)solvable) radical [... up to isomorphisms].

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In this talk

Lie algebras of nilradical a free nilpotent Lie algebra of type 2. [a]

[a] Pilar Benito and Daniel De-la-Concepción (2013):
A note on extensions of nilpotent Lie algebras of type 2
arXiv:1307.8419

Related preliminary results

Among others (. . .)

- General context (∗)
  - L. Auslander, J. Brezin (1968)[2]
    Malcev decompositions, i.e.,Levi decomposition focus on nilradical.
  - A.L. Onishchick, Y.B. Khakimdzhanov (1975)[3]
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- Restricted context

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[3] On semidirect sums of Lie algebras, Mat. Zametki 18 (1975), 31-40. (Math. Notes 18 (1976), 600-604.)

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Related preliminary results

- Recent results

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  Nilradical a nilpotent Lie algebra of type 2 or 3 and of maximal nilindex; in low nilindex they are free nilpotent.

  J.M. Ancochea-Bermúdez, R. Campoamor-Stursberg, L. García-Vergnolle (2011)[6]

  As previous one but adding non-solvable extensions.

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Examples and features

- Heisenberg algebras[8] $\mathfrak{h}_{2n+1} (n \geq 1)$:

$$\mathrm{Der}\, \mathfrak{h}_{2n+1} = \mathfrak{sp}_{2n}(V, b) \oplus \mathfrak{n}, \mathfrak{n} \text{ nilpotent radical}$$

[R&W, 1993]

The classification of solvable algebras with nilradical $\mathfrak{h}_{2n+1}$ boils down to a classification of abelian subalgebras of the symplectic algebra $\mathfrak{sp}_{2n}(V, b)$.
For $n = 1, 2$, real and complex fields: complete classification.

The Non-solvable case? Open.

[8] Center one dimensional and equal to $\mathfrak{h}_{2n+1}^2 = kz$, $\mathfrak{h}_{2n+1} = V \oplus kz$.

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- Filiform $\mathfrak{f}_n, n \geq 3$ and Quasifliform $\mathfrak{q}_n, n \geq 4$ algebras[9]

$$\dim \mathfrak{f}_n/(\mathfrak{f}_n)^2 = 2 \text{ and } \dim \mathfrak{q}_n/(\mathfrak{q}_n)^2 = 2, 3$$

$\Longrightarrow$ Classification encoded by free nilpotent of types $2, 3$;

$\mathfrak{f}_3 \cong \mathfrak{h}_3$ and $\mathfrak{q}_5$ are free nilpotent of type 2.

[9] Defined as nilpotent Lie algebras with dimension n and nilindex $n-1$ ($\mathfrak{f}$-case) or $n-2$ ($\mathfrak{q}$-case).

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Examples and features

The derivation algebra of $\mathfrak{f}_3$ and $\mathfrak{q}_5$ have the simple split 3-dimensional algebra $\mathfrak{sl}_2(k)$ as Levi subalgebra:
Exceptional feature :-)).

For sizes $n \neq 3$, $m \neq 5$, $\mathrm{Der}\,\mathfrak{f}_n$ and $\mathrm{Der}\,\mathfrak{q}_m$ are solvable. So:

Only solvable Lie algebras can have filiform or quasifiliform nilradicals (sizes different from 3 and 5), if we are looking for nontrivial decompositions.

[AB&CS&GV, 2011]

Classify Lie algebras with nilradical a "natural graded" filiform or quasifiliform algebra.
(Some errors in the classification)

Non-"natural graded"? Open.

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Terminology

$(\mathfrak{n}, [a, b])$ nilpotent Lie algebra ($\mathfrak{n}^{t+1} = 0$),

Lower central series:

$\mathfrak{n}, \mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}], \mathfrak{n}^3 = [\mathfrak{n}, \mathfrak{n}^2], \ldots, \mathfrak{n}^{j+1} = [\mathfrak{n}, \mathfrak{n}^j]$;

Nil-index: smallest t such that $\mathfrak{n}^{t+1} = 0$;

type: codimension of $\mathfrak{n}^2$ in $\mathfrak{n}$;

$\mathrm{Der}\,\mathfrak{n}$: derivation Lie algebra;

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$\mathfrak{n}, \mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}], \mathfrak{n}^3 = [\mathfrak{n}, \mathfrak{n}^2], \ldots, \mathfrak{n}^{j+1} = [\mathfrak{n}, \mathfrak{n}^j]$;

Nil-index: smallest t such that $\mathfrak{n}^{t+1} = 0$;

type: codimension of $\mathfrak{n}^2$ in $\mathfrak{n}$;

$\mathrm{Der}\,\mathfrak{n}$: derivation Lie algebra;

$\mathrm{Aut}\,\mathfrak{n}$: automorphism group.

Free nilpotent definition

Let $\mathfrak{FL}(\mathfrak{m})$ be the free Lie algebra on $\mathfrak{m} = \{x_1, \dots, x_d\}$,

For any $t \geq 1$, the quotient

$$\mathfrak{n}_{d,t} = \frac{\mathfrak{FL}(\mathfrak{m})}{\mathfrak{FL}(\mathfrak{m})^{t+1}}$$

is call free *t*-nilpotent Lie algebra on *d*-generators.

Free nilpotent basic facts

1. Type $\mathfrak{n}_{d,t} = d$ and $\mathfrak{n}_{d,t}$ is t-nilpotent.
2. Graduated: $\mathfrak{n}_{d,t} = \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots \oplus \mathfrak{m}^t$.
3. Any t-nilpotent and d-type is a quotient of $\mathfrak{n}_{d,t}$.
4. For a given quotient

$$\mathfrak{n} = \frac{\mathfrak{n}_{d,t}}{J}, \text{ we have } \operatorname{Der} \mathfrak{n} = \frac{\mathcal{D}_J}{\mathcal{D}_0},$$

   where $\mathcal{D}_J$ is the subalgebra of J-invariant derivations and $\mathcal{D}_0$ is the set of derivations which send $\mathfrak{n}_{d,t}$ onnto J [a].
5. $\operatorname{Der} \mathfrak{n}_{d,t} = \mathfrak{sl}_d(k) \oplus \mathfrak{r}$, $\mathfrak{r}$ solvable radical[10].
6. $\operatorname{Aut} \mathfrak{n}_{d,t} \cong GL_d(k) \ltimes H$ (semidirect product).

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The derivation Lie algebra

$$\operatorname{Der}\mathfrak{n}_{d,t} = \bigoplus_{j=1}^{t} \operatorname{Der}_j \mathfrak{n}_{d,t},$$

where $\operatorname{Der}_j \mathfrak{n}_{d,t} = \{d \in \operatorname{Der}\mathfrak{n}_{d,t} : d(\mathfrak{m}) \subseteq \mathfrak{m}^j\}$.

1. $[d_i, d_s] \in \operatorname{Der}_{i+s-1} \mathfrak{n}_{d,t}$;
2. $\operatorname{Der}_1 \mathfrak{n}_{d,t} = \operatorname{Der}_1^0 \mathfrak{n}_{d,t} \oplus k \cdot id_{d,t} \cong \mathfrak{gl}_d(k)$;
3. $id_{d,t}\,|_{\mathfrak{m}^s} = s \cdot Id$; $[id_{d,t}, d_s] = (s-1)d_s$ for any $d_s \in \operatorname{Der}_s \mathfrak{n}_{d,t}$.
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First fast result

According to Malcev's 44, the bigest faithful Lie algebra of characteristic zero with solvabale radical $\mathfrak{n}_{d,t}$ is:

$$\mathfrak{g}_{d,t} = \mathrm{Der}_1^0\, \mathfrak{n}_{d,t} \oplus_{id} \mathfrak{n}_{d,t}.$$

Even more:

Any faithful Lie algebra of characteristic zero with solvable radical $\mathfrak{n}_{d,t}$ is isomorphic to a Lie algebra of the form $\mathfrak{s} \oplus_{id} \mathfrak{n}_{d,t}$ where $\mathfrak{s}$ is either the null algebra or a semisimple subalgebra of $Der_1^0\, \mathfrak{n}_{d,t} \cong \mathfrak{sl}_d(k)$ (simple Lie algebra of Cartan Type A_{d-1}). In particular, $\mathfrak{n}_{2,t}$, $\mathfrak{g}_{2,t}$ and the direct sum as ideals $\mathfrak{s} \oplus \mathfrak{n}_{2,t}$ and $\mathfrak{s} \oplus \mathfrak{g}_{2,t}$, where $\mathfrak{s}$ is semisimple, exhaust the list of Lie algebras with solvable radical $\mathfrak{n}_{2,t}$.

Benito&de-la-Concepción, 2013

Let $\mathfrak{r}$ be a solvable Lie algebra with nilradical $\mathfrak{n}_{2,t}$. Then, $\mathfrak{r} = \mathfrak{n}_{2,t} \oplus \mathfrak{t}$ where $\mathfrak{t}$ is a vector space equidimensional to the projection $\mathfrak{p}_1(\mathfrak{t}) = \mathrm{proj}_{\mathrm{Der}_1\, \mathfrak{n}_{2,t}} \mathrm{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t}$ which is an abelian subalgebra of $\mathrm{Der}_1\, \mathfrak{n}_{2,t}$ of dimension at most 2 and the derivation set $\mathrm{ad}_{\mathfrak{n}_{2,t}} \mathfrak{t}$ contains no nilpotent elements. Moreover, in case $\mathfrak{t} = k \cdot x \oplus k \cdot y$, $\mathfrak{p}_1(\mathfrak{t}) = C_{\mathrm{Der}_1\, \mathfrak{n}_{2,t}}(\delta)$ for some $\delta \in \mathrm{Der}_1\, \mathfrak{n}_{2,t} \backslash k \cdot id_{2,t}$ and the braket derivation $[\mathrm{ad}_{\mathfrak{n}_{2,t}} x, \mathrm{ad}_{\mathfrak{n}_{2,t}} y]$ is an inner derivation of $\mathfrak{n}_{2,t}$.

In other words: only 1-solvable $\mathfrak{n}_{2,t} \oplus k \cdot x$, and 2-solvable $\mathfrak{n}_{2,t} \oplus k \cdot x \oplus k \cdot y$ extensions are possible; the extensions are given by derivations with some restrictions which are controled by the subalgebra $\mathrm{Der}_1\, \mathfrak{n}_{2,t} = \mathfrak{sl}_2(k) \oplus k \cdot id_{2,t}$.

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Let $\mathfrak{g}$ be a non-solvable Lie algebra with nilradical $\mathfrak{n}_{2,t}$. Then $\mathfrak{g}$ is one of the following Lie algebras:

a) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ is a direct sum as ideals of a semisimple algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ with nilradical $\mathfrak{n}_{2,t}$;

b) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}_{2,t}$, a direct sum as ideals where $\mathfrak{s}$ is either the null algebra or any arbitrary semisimple algebra or

c) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}(\delta)$, a direct sum as ideals where $\mathfrak{s}$ is either the null algebra or any arbitrary semisimple algebra and $\mathfrak{g}(\delta)$ is the Lie algebra $\mathfrak{g}(\delta) = \mathfrak{g}_{2,t} \oplus k \cdot x$, with $[\mathrm{Der}_1^0 \mathfrak{n}_{2,t}, x] = 0$ and $ad_{\mathfrak{n}_{2,t}} x = id_{2,t} + \delta$, $\delta \in \mathfrak{N}_{2,t}$ where $\delta \in C_{\mathrm{Der}\,\mathfrak{n}_{2,t}}(\mathrm{Der}_1^0 \mathfrak{n}_{2,t})$.

So: the algebras in a) depends on solvable extensions and those in c) are given by derivations that belong to the trivial module of $\mathrm{Der}\,\mathfrak{n}_{2,t}$ with respect to their Levi subalgebra.

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Standar basis for $\mathfrak{n}_{2,t}$

The free algebra $\mathfrak{n}_{2,t}$ is determine by the $\mathfrak{sl}_2(k)$-module $\mathfrak{m}$:

$\mathfrak{m}$	$\mathfrak{m}^2$	$\mathfrak{m}^3$	$\mathfrak{m}^4$	$\mathfrak{m}^5$
$V(1)$	$V(0)$	$V(1)$	$V(2)$	$V(3) \oplus V(1)$
v_0, v_1	$[v_0, v_1] = w_0$	$[v_0, w_0] = z_0$	$[v_0, z_0] = x_0$	$\dots$
		$[v_0, w_0] = z_1$	$[v_0, z_1] = [v_1, z_0] = \frac{1}{2}x_1$	$\dots$
			$[v_1, z_1] = x_2$	

Special feature: Nested basis

$$\mathfrak{n}_{2,1} = V(1)$$
$$\mathfrak{n}_{2,2} = V(1) \oplus V(0)$$
$$\mathfrak{n}_{2,3} = V(1) \oplus V(0) \oplus V(1)$$
$$\dots$$

Der $\mathfrak{n}_{2,3}$:

in blue, $\mathrm{Der}_1\, \mathfrak{n}_{2,3}$ isomorphic to $\mathfrak{gl}_2(k)$

$$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 2\beta & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_5 & \alpha_1+3\beta & \alpha_2 \\ \alpha_8 & \alpha_9 & -\alpha_4 & \alpha_3 & -\alpha_1+3\beta \end{pmatrix}$$

Aut $\mathfrak{n}_{2,3}$:

$\epsilon = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$; in blue, $GL_2(k)$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 \\ \alpha_5 & \alpha_6 & \epsilon & 0 & 0 \\ \alpha_7 & \alpha_8 & \alpha_1\alpha_6-\alpha_2\alpha_5 & \epsilon\alpha_1 & \epsilon\alpha_2 \\ \alpha_9 & \alpha_{10} & \alpha_3\alpha_6-\alpha_4\alpha_5 & \epsilon\alpha_3 & \epsilon\alpha_4 \end{pmatrix}$$

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$\epsilon = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$; in blue, $GL_2(k)$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 \\ \alpha_5 & \alpha_6 & \epsilon & 0 & 0 \\ \alpha_7 & \alpha_8 & \alpha_1\alpha_6-\alpha_2\alpha_5 & \epsilon\alpha_1 & \epsilon\alpha_2 \\ \alpha_9 & \alpha_{10} & \alpha_3\alpha_6-\alpha_4\alpha_5 & \epsilon\alpha_3 & \epsilon\alpha_4 \end{pmatrix}$$

Der $\mathfrak{n}_{2,1}$, Der $\mathfrak{n}_{2,2}$ and Der $\mathfrak{n}_{2,3}$: Cut and Paste descriptions

	$\alpha_1 + \beta$	α_2	0	0	0
$\mathfrak{n}_{2,1}$	α_3	$-\alpha_1 + \beta$	0	0	0
$\mathfrak{n}_{2,2}$	α_4	α_5	2β	0	0
	α_6	α_7	α_5	$\alpha_1 + 3\beta$	α_2
$\mathfrak{n}_{2,3}$	α_8	α_9	$-\alpha_4$	α_3	$-\alpha_1 + 3\beta$

1-solvable extensions' map

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$\mathfrak{n}_{2,1}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
	$\mathfrak{n}_{2,3}$	$\alpha{=}{-1} \updownarrow \; \updownarrow \alpha{=}{-1}$
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\mathfrak{n}_{2,2}$ $\overset{\alpha=0}{\longleftrightarrow}$ $\overset{\alpha=0}{\longleftrightarrow}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 & 0 \\ 0 & 0 & 0 & 2+\alpha & 0 \\ 0 & 0 & 0 & 0 & 1+2\alpha \end{pmatrix}$

\# 1-extensions for $\mathfrak{n}_{2,1}$ and $\mathfrak{n}_{2,2} = 2$;
\# 1-extensions for $\mathfrak{n}_{2,3} = 4$.

1-solvable extensions' map

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$\mathfrak{n}_{2,1}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
	$\mathfrak{n}_{2,3}$	$\alpha=-1 \updownarrow \quad \updownarrow \alpha=-1$
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\mathfrak{n}_{2,2}$ $\overset{\alpha=0}{\longleftrightarrow}$ $\overset{\alpha=0}{\longleftrightarrow}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 & 0 \\ 0 & 0 & 0 & 2+\alpha & 0 \\ 0 & 0 & 0 & 0 & 1+2\alpha \end{pmatrix}$

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2-solvable extensions' map

	$\mathfrak{n}_{2,1}$	
	$\mathfrak{n}_{2,3}$	
	$\mathfrak{n}_{2,2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

$\#$ 1-extensions for $\mathfrak{n}_{2,1}$, $\mathfrak{n}_{2,2}$ and $\mathfrak{n}_{2,3} = 1$.
Even more: Unique extension given by a CSA of $\mathrm{Der}_1 \mathfrak{n}_{2,t} \cong \mathfrak{gl}_2(k)$.

2-solvable extensions' map

	$\mathfrak{n}_{2,1}$	
	$\mathfrak{n}_{2,3}$	
	$\mathfrak{n}_{2,2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

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Even more: Unique extension given by a CSA of $\mathrm{Der}_1 \mathfrak{n}_{2,t} \cong \mathfrak{gl}_2(k)$.

Non-solvable extensions of type $\mathfrak{g}(\delta)$ (item c) 2nd main result)

The centralizer $C_{\mathrm{Der}\,\mathfrak{n}_{2,t}}(\mathrm{Der}_1^0\,\mathfrak{n}_{2,t})$:

- $t = 1, 2$: $\quad k \cdot id_{2,t}$
- $t = 3$: $C_{\mathrm{Der}\,\mathfrak{n}_{2,t}}(\mathrm{Der}_1^0\,\mathfrak{n}_{2,t}) = k \cdot id_{2,t} \oplus k \cdot \mathrm{ad}\,\omega_0$, where $\omega_0 = [v_0, v_1]$.

Up to isomorphisms, $\delta = id_{2,t}$, so

$$\mathfrak{g}(id_{2,t}) = \mathfrak{g}_{2,t} \oplus k \cdot id_{2,t}$$

is the unique possibility for $t = 1, 2, 3$.

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