# Triality

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triality: There is an $S_3$-action and ...

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$$\begin{array}{lll} d^\sigma = d, & \mathrm{L}_a^\sigma = -\mathrm{R}_a, & \mathrm{R}_a^\sigma = -\mathrm{L}_a \\ d^\rho = d, & \mathrm{L}_a^\rho = \mathrm{R}_a, & \mathrm{R}_a^\rho = -\mathrm{L}_a - \mathrm{R}_a \end{array}$$

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Then $\mathrm{L}_a^\sigma - \mathrm{L}_a = -\mathrm{R}_a - \mathrm{L}_a$, $(\mathrm{L}_a^\sigma - \mathrm{L}_a)^\rho = \mathrm{L}_a$, $(\mathrm{L}_a^\sigma - \mathrm{L}_a)^{\rho^2} = \mathrm{R}_a$
so sum $= 0$.

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Results on groups with triality via connections with loops:

- Glauberman ('68)
- Doro ('78)
- Grishkov-Zavarnitsine ('06)
- Hall ('10)
- B,M,P-I ('13)

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J. Hall, *Moufang Loops and Groups with Triality are Essentially the Same Thing*

Lie algebras with triality

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$\mathcal{L}(\mathfrak{m})$: Lie algebra generated by symbols $\ell_x, \mathrm{r}_x,$ for $x \in \mathfrak{m}$ s.t. $x \mapsto \ell_x,\ \ x \mapsto \mathrm{r}_x$ are bilinear and

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# The Map II

$\mathcal{L}(\mathfrak{m}) \leftarrow\!\shortmid \mathfrak{m}$

Lie algebras with triality $\rightleftarrows$ Malcev algebras

$G \mapsto \mathcal{M}(G)$

Groups with triality $\rightleftarrows$ Moufang loops

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$\mathbb{F}G$ is a Hopf algebra with triality with $T(g) = g^{\sigma} g^{-1}$.

** Replace $T$ with $T'(g) = T(\sigma(g^{-1})) = g^{-1} g^{\sigma}$ to get earlier defn.*

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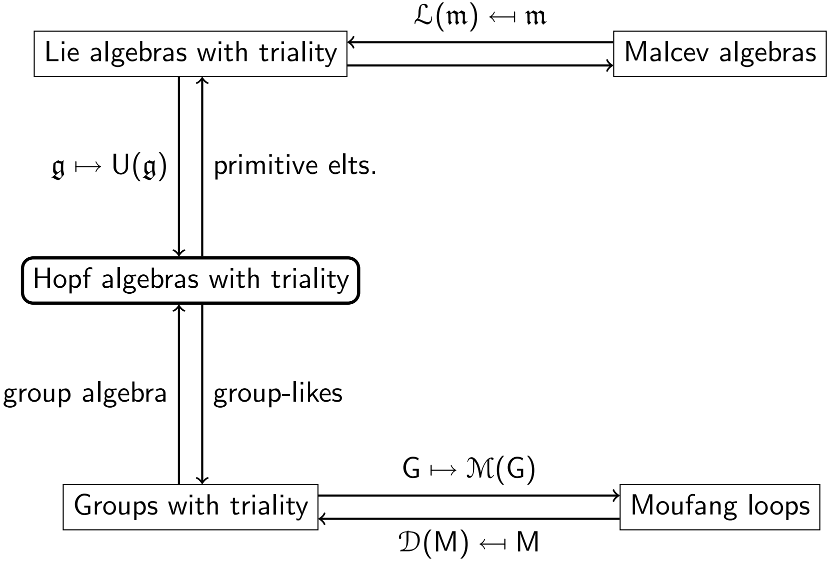
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- In this case, say $(\mathrm{U}, \Delta, \epsilon)$ is a Moufang-Hopf algebra.

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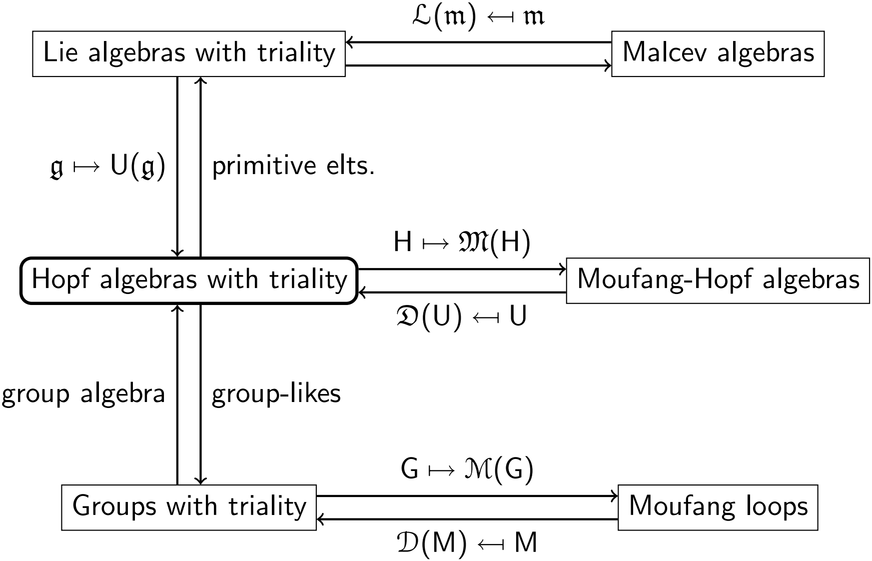
**Thm.** (B,M,P-I) $\mathfrak{D}(\mathrm{U})$ with $\rho$ (above) and $\sigma$ given by $\mathrm{P}_u \xrightarrow{\sigma} \mathrm{P}_{S(u)}, \quad \mathrm{L}_u \xrightarrow{\sigma} \mathrm{R}_{S(u)}, \quad \mathrm{R}_u \xrightarrow{\sigma} \mathrm{L}_{S(u)}$ is a Hopf algebra with triality.

# The Map IV



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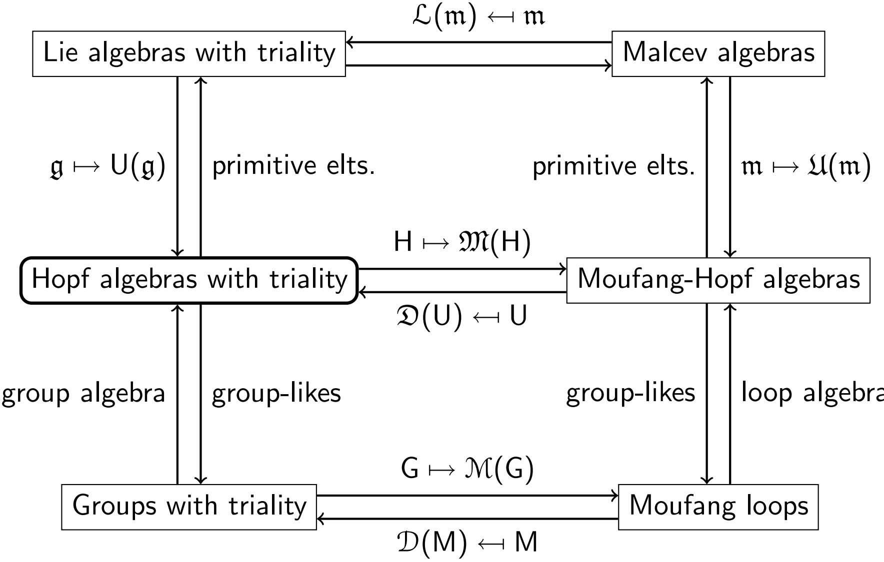

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# Connections with Universal Enveloping Algebras

- **Thm.** (B,M,P-I) For any U cocommutative Moufang-Hopf algebra,
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- **Remark.** (Shestakov–Pérez-Izquierdo ('04))
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Thm. (Madariaga) N is a non-cocomm. Hopf alg. with triality.