On Loday's Parametrized One-Relation Algebras

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Introduction

Loday introduced the **parametrized one-relation algebras** determined by six scalars $x_1, \ldots, x_6$ from the base field F. Any such scalars determine a variety of algebras over F with one bilinear operation satisfying Loday's polynomial identity

$$(ab)c \equiv$$
$$x_1 a(bc) + x_2 a(cb) + x_3 b(ac) + x_4 b(ca) + x_5 c(ab) + x_6 c(ba),$$

which can be written more compactly as

$$(ab)c \equiv \sum_{\sigma \in S_3} x_\sigma a^\sigma (b^\sigma c^\sigma),$$

That is, any monomial with the association type $(--)-$ can be expressed as a linear combination of permutations of monomials with the association type $-(--)$.

Some familiar varieties of algebras have this form:

- **Associative**: $(ab)c \equiv a(bc)$, $(x_1, \dots x_6) = (1, 0, 0, 0, 0, 0)$
- **Leibniz**: $a(bc) \equiv (ab)c + b(ac)$ (derivation rule), equivalently $(ab)c \equiv a(bc) - b(ac)$, $(x_1, \dots x_6) = (1, 0, -1, 0, 0, 0)$
- **Zinbiel**: $(ab)c \equiv a(bc) + a(cb)$, $(x_1, \dots x_6) = (1, 1, 0, 0, 0, 0)$

Poisson algebras, usually defined with two operations, can also be defined with only one operation: a surprising fact discovered by Livernet-Loday (unpublished) and Markl-Remm (2006). The usual definition involves a commutative associative product $a \cdot b$, and a Lie bracket $[a, b]$, related by the derivation rule,

$$[a, b \cdot c] \equiv [a, b] \cdot c + b \cdot [a, c].$$

"Depolarization" replaces these operations by $ab = a \cdot b + [a, b]$, satisfying Loday's identity with $(x_1, \dots x_6) = (1, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, 0)$:

$$(ab)c \equiv a(bc) + \tfrac{1}{3}\big[\, a(cb) - b(ac) + b(ca) - c(ab) \,\big].$$

Free algebras in these four varieties satisfy **Property S**: *For all $n \geq 1$, the free algebra on n generators is isomorphic as a graded vector space to the tensor algebra on n generators.*

That is, a basis for the free algebra on $A = \{a_1, \ldots, a_n\}$ consists of the free monoid A^*; the multiplication depends on the variety $\mathcal{V}$:

- **Associative**: the product is defined by concatenation in A^*:

$$(a_{i_1} \cdots a_{i_p})(a_{i_{p+1}} \cdots a_{i_{p+q}}) = a_{i_1} \cdots a_{i_{p+q}}.$$

- **Leibniz**: Loday-Pirashvili (1993) showed that the product is determined by the equation $[a_{i_1} \cdots a_{i_p}, a_{i_{p+1}}] = a_{i_1} \cdots a_{i_p} a_{i_{p+1}}$. That is, we define $[x, y]$ inductively on the degree q of y, using the last equation for $q = 1$ and the derivation rule for $q \geq 2$:

$$\begin{aligned}
&[a_{i_1} \cdots a_{i_p}, a_{i_{p+1}} \cdots a_{i_{p+q+1}}] = \\
&\qquad [[a_{i_1} \cdots a_{i_p}, a_{i_{p+1}} \cdots a_{i_{p+q}}], a_{i_{p+q+1}}] \\
&\qquad\qquad - [[a_{i_1} \cdots a_{i_p}, a_{i_{p+q+1}}],\ a_{i_{p+1}} \cdots a_{i_{p+q}}].
\end{aligned}$$

- **Zinbiel**: Loday (1995) showed that the product is "one-half" of the shuffle product on monomials; that is, we have

$$(a_{i_1} \cdots a_{i_p})(a_{i_{p+1}} \cdots a_{i_{p+q}}) = (1 \otimes \mathrm{sh}_{p-1,q})(a_{i_1} \cdots a_{i_{p+q}})$$

  where $\mathrm{sh}_{p-1,q}$ is the sum over all $(p-1, q)$-shuffles permuting the positions (not the subscripts) of $a_{i_2}, \ldots, a_{i_{p+q}}$. That is,

$$\sigma(2) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(p+q).$$

- **Poisson**: The free Poisson algebra on n generators is linearly isomorphic to the symmetric algebra over the free Lie algebra on n generators. The free associative algebra is the universal envelope of the free Lie algebra. The PBW theorem shows that a monomial basis for the free associative algebra consists of monomials in an ordered basis of the free Lie algebra, and that the associated graded algebra is isomorphic to the (commutative) polynomial algebra over the free Lie algebra.

We can reformulate **Property S** and say that the operad defining the variety $\mathcal{V}$ is isomorphic to the associative operad: *For all $n \geq 1$, the multilinear subspace of degree n in the free algebra on n generators is isomorphic to the regular representation of the symmetric group S_n (the group algebra with the left action of S_n).*

Open problem: Determine all values of $x_1, \ldots, x_6$ for which the parametrized one-relation algebras satisfy Property S.

Livernet-Loday studied a one-parameter family of solutions interpolating between associative and Poisson algebras. We have discovered three new one-parameter families of solutions; in each family, the corresponding algebraic structures are isomorphic:

1. the family $(ab)c \equiv \lambda c(ab)$ $(\lambda \neq \pm 1)$;
2. a one-parameter family including the Leibniz operad;
3. the Koszul dual family including the Zinbiel operad.

Conjecture: Every solution belongs to one of these 4 families.

Livernet-Loday's work on Poisson algebras

Livernet-Loday considered associative algebras with product ab, and "polarized" this product into the commutative Jordan product $a \circ b = ab + ba$ and anticommutative Lie bracket $[a, b] = ab - ba$. They showed that associativity of ab is equivalent to two identities relating the (skew-)symmetric operations $a \circ b$ and $[a, b]$:

$$[a, b \circ c] \equiv [a, b] \circ c + b \circ [a, b], \quad (a \circ b) \circ c - a \circ (b \circ c) \equiv [b, [a, c]].$$

The first is the derivation rule; the second says that the Jordan associator is the Lie triple product. Livernet-Loday then considered a deformation with three identities involving a parameter q:

$$\left.\begin{aligned} &[a, b \circ c] \equiv [a, b] \circ c + b \circ [a, b], \\ &(a \circ b) \circ c - a \circ (b \circ c) \equiv q\,[b, [a, c]], \\ &[a, [b, c]] + [b, [c, a]] + [c, [a, b]] \equiv 0. \end{aligned}\right\} \qquad (LL_q)$$

(If $q \neq 0$, then the second identity implies the third.)

If $q = 0$ then the second identity is associativity for the $\circ$ product, and we obtain the two-operation definition of Poisson algebras.

If $q = 1$ then we obtain the polarized form of associativity.

If we depolarize the identities (LL_q) then we obtain algebras with one operation satisfying this identity for $q \neq -3$,

$$(ab)c \equiv a(bc) + \frac{1-q}{3+q}\Big[\, a(cb) - b(ac) + b(ca) - c(ab) \,\Big],$$

and a slightly different variety for $q = -3$.

These are all examples of parametrized one-relation algebras.

For $q \neq 0$, the depolarization of (LL_q) is equivalent to associativity.

We have a one-parameter family which gives Poisson algebras for $q = 0$ and associative algebras for $q \neq 0$.

(These details are taken from Markl-Remm (2006).)

The family $(ab)c \equiv \lambda c(ab)$ for $\lambda \neq \pm 1$

Our first new family of solutions is defined by a very simple identity.

Theorem

For $\lambda \neq \pm 1$, the identity $(ab)c \equiv \lambda c(ab)$ satisfies condition S.

Proof: For $n \leq 3$, it is clear that the multilinear subspace of degree n in the free algebra on n generators satisfying $(ab)c \equiv \lambda c(ab)$ for any λ is isomorphic to the regular representation of S_n.

For $n = 4$, applying the identity twice gives

$$(ab)(cd) \equiv \lambda(cd)(ab) \equiv \lambda^2(ab)(cd).$$

Hence $(1-\lambda^2)\,(ab)(cd) \equiv 0$, and $\lambda \neq \pm 1$ implies $(ab)(cd) \equiv 0$. Thus the product of two decomposable monomials is zero, and it remains to determine the product of a monomial with a generator.

Applying the identity to the association types in degree 4 gives

$$
\begin{aligned}
((ab)c)d &\equiv \lambda^2 d(c(ab)), \\
(a(bc))d &\equiv \lambda d(a(bc)), \\
a((bc)d) &\equiv \lambda a(d(bc)).
\end{aligned}
$$

Thus any monomial in degree 4 is a scalar multiple (possibly 0) of a monomial with association type $-(-(--))$. Hence the multilinear subspace is isomorphic to the regular representation of S_4.

Now assume that $n \geq 4$ and that any monomial of degree n is a scalar multiple of a monomial x which is in *normal form*: that is, a sequence of left multiplications applied to a generator. For degree $n+1$, the monomial ax is already in normal form, and $xa = \lambda ax$ is a scalar multiple of a monomial in normal form. By induction, every monomial is a scalar multiple of a monomial in normal form. Furthermore, the $n!$ multilinear monomials in normal form are linearly independent, and span the regular representation of S_n.

As an immediate consequence, we obtain the multiplication formula for monomials in the free algebra.

Corollary

Let $A(X)$ be the free algebra generated by the set X subject to the identity $(ab)c \equiv \lambda c(ab)$ where $\lambda \neq \pm 1$. A basis for A consists of the monomials which can be expressed as a sequence (possibly empty) of left multiplications applied to a generator. If x, y are two such monomials, then their product is as follows:

$$xy = \begin{cases} 0 & \textit{if } \deg(x) > 1 \textit{ or } \deg(y) > 1, \\ ay & \textit{if } \deg(x) = 1 \textit{ and } x = a \in X, \\ \lambda bx & \textit{if } \deg(y) = 1 \textit{ and } y = b \in X. \end{cases}$$

For $\lambda = \pm 1$, the multilinear subspaces for degrees $n = 4, 5, 6$ have dimensions 36, 300, 3240 (not 24, 120, 720). It is an open problem to determine the structure of the free algebras in these two cases.

A computational search for new solutions

We write Loday's parametrized identity as $I(a,b,c) \equiv 0$, assuming that $x_0, \dots, x_6$ are integers and $x_0 \geq 1$, where $I(a,b,c)$ is

$$x_0(ab)c - x_1 a(bc) - x_2 a(cb) - x_3 b(ac) - x_4 b(ca) - x_5 c(ab) - x_6 c(ba).$$

Every multilinear identity in degree 4 which follows from $I \equiv 0$ is in the S_4-module generated by the identities $J_k \equiv 0$ $(k = 1, \dots, 5)$:

$$\begin{aligned} J_1 &= I(ad,b,c), & J_2 &= I(a,bd,c), & J_3 &= I(a,b,cd), \\ J_4 &= I(a,b,c)d, & J_5 &= dI(a,b,c). \end{aligned}$$

Applying all permutations of a,b,c,d gives 120 identities which span the S_4-module of consequences of I in degree 4. Each term consists of a permutation of a,b,c,d and an association type:

$$((--)-)-, \ \ (-(--))-, \ \ (--)(--), \ \ -((--)-), \ \ -(-(--)).$$

We construct a 120×120 matrix L in which the (i, j) entry is the coefficient of the j-th monomial in the i-th identity.

- The entries of L are quadratic polynomials in $x_0, \ldots, x_6$.
- The row space of L is the S_4-module of consequences of I.
- The nullspace of L is the multilinear subspace of degree 4 in the free algebra on 4 generators satisfying $I \equiv 0$.

We considered all $x_0 \in \{1, \ldots, 5\}$ and $x_1, \ldots, x_6 \in \{-5, \ldots, 5\}$; altogether 8 857 805 cases; but we excluded $\gcd(x_0, \ldots, x_6) \neq 1$.

We used the computer algebra system Maple to calculate the structure of the nullspace as a direct sum of simple S_4-modules labelled by the partitions 4, 31, 22, 211, 1111.

We found that the nullspace has dimension 24 in 141 cases, and is isomorphic to the regular representation in 83 cases. Thus there are 58 "irregular" cases for which the S_4-module has dimension 24 but is not isomorphic to the regular representation.

The 83 cases which produce the regular representation of S_4 are:

1. 37 cases in the family $(ab)c \equiv \lambda c(ab)$ where $\lambda \neq \pm 1$.
2. 38 cases in the Loday-Livernet family with $\lambda \neq -1$ $(q \neq \infty)$:

$$(ab)c \equiv a(bc) + \lambda\big[a(cb) - b(ac) + b(ca) - c(ab)\big].$$

3. The (left) Leibniz identity, $(ab)c \equiv a(bc) - b(ac)$.
4. The (right) Zinbiel identity, $(ab)c \equiv a(bc) + a(cb)$.
5. The following 6 identities:

$$\begin{aligned}
(ab)c &\equiv \tfrac{1}{3}[\,5\,a(bc)+5\,a(cb)+b(ac)+b(ca)+c(ab)+2\,c(ba)\,]\\
(ab)c &\equiv \tfrac{1}{3}[\,5\,a(bc)-a(cb)-5\,b(ac)+b(ca)+c(ab)-2\,c(ba)\,]\\
(ab)c &\equiv \tfrac{1}{3}[\,a(bc)+a(cb)-4\,b(ac)-4\,b(ca)-4\,c(ab)-2\,c(ba)\,]\\
(ab)c &\equiv \tfrac{1}{3}[\,a(bc)+4\,a(cb)-b(ac)-4\,b(ca)-4\,c(ab)+2\,c(ba)\,]\\
(ab)c &\equiv \tfrac{1}{3}[\,5\,a(bc)+5\,a(cb)-4\,b(ac)-4\,b(ca)-4\,c(ab)+2\,c(ba)\,]\\
(ab)c &\equiv \tfrac{1}{3}[\,5\,a(bc)+4\,a(cb)-5\,b(ac)-4\,b(ca)-4\,c(ab)-2\,c(ba)\,]
\end{aligned}$$

The 58 cases for which the S_4-module has dimension 24 but is not the regular representation belong to four one-parameter families, which can be written as follows:

$$(ab)c - c(ba) \equiv \lambda \sum_{\sigma \in S_3} a^\sigma (b^\sigma c^\sigma),$$

$$(ab)c - c(ba) \equiv \lambda \sum_{\sigma \in S_3} \epsilon(\sigma)\, a^\sigma (b^\sigma c^\sigma),$$

$$(ab)c - c(ab) \equiv \lambda \sum_{\sigma \in S_3} a^\sigma (b^\sigma c^\sigma),$$

$$(ab)c - c(ab) \equiv \lambda \sum_{\sigma \in S_3} \epsilon(\sigma)\, a^\sigma (b^\sigma c^\sigma).$$

These "irregular" cases, with the "right" dimension but the "wrong" representation in degree 4, do not satisfy Property S, but they deserve further consideration.

Koszul duality

Koszul duality was defined for associative algebras with quadratic relations by Priddy (1970), and extended to quadratic operads by Ginzburg-Kapranov (1994).

An operad is quadratic if the identities defining the corresponding variety of algebras are quadratic in the operations; thus, for binary operations we require identities of degree 3, as in the familiar examples of associative, Leibniz, Zinbiel, and Poisson algebras.

Associative algebras can be regarded as monoids in the category of vector spaces. Similarly, one of the monoidal structures on the category of $\mathbb{S}$-modules allows us to regard an operad as a monoid in this category. (An $\mathbb{S}$-module is a sequence $[\, M(n) \mid n \geq 1 \,]$ of modules over the symmetric groups S_n.)

The connection between operads and algebras is as follows: The S_n-module $M(n)$ can be identified with the multilinear subspace of degree n in the free algebra on n generators.

In the simplest case of a single binary operation, a quadratic operad is determined by the relations in an S_3-submodule R of the multilinear subspace of degree 3 in the free nonassociative algebra, which has dimension 12 and basis $\{a^\sigma b^\sigma \cdot c^\sigma, a^\sigma \cdot b^\sigma c^\sigma \mid \sigma \in S_3\}$. The Koszul dual is determined by the relations in the orthogonal complement $R^\perp$ with respect to this symmetric bilinear form:

$$\begin{array}{ll} \text{(type 1)} & \langle\, a^\sigma b^\sigma \cdot c^\sigma,\ a^\tau b^\tau \cdot c^\tau \,\rangle = \delta_{\sigma\tau}\, \epsilon(\sigma), \\ \text{(type 2)} & \langle\, a^\sigma \cdot b^\sigma c^\sigma,\ a^\tau \cdot b^\tau c^\tau \,\rangle = -\delta_{\sigma\tau}\, \epsilon(\sigma), \\ \text{(mixed)} & \langle\, a^\sigma b^\sigma \cdot c^\sigma,\ a^\tau \cdot b^\tau c^\tau \,\rangle = 0. \end{array}$$

Loday noted that the Koszul dual of the operad defined by

$$ab \cdot c \equiv \sum_{\sigma \in S_3} x_\sigma\, a^\sigma \cdot b^\sigma c^\sigma,$$

is given by the following identity:

$$a \cdot bc \equiv \sum_{\sigma \in S_3} \epsilon(\sigma)\, x_\sigma\, a^{\sigma^{-1}} b^{\sigma^{-1}} \cdot c^{\sigma^{-1}}.$$

Start with the original parametrized relation:

$$ab \cdot c \equiv x_1 a \cdot bc + x_2 a \cdot cb + x_3 b \cdot ac + x_4 b \cdot ca + x_5 c \cdot ab + x_6 c \cdot ba$$

Write it in terms of permutations of positions (not variables):

$$() \equiv x_1() + x_2(23) + x_3(12) + x_4(132) + x_5(123) + x_6(13)$$

Replace each permutation by its inverse:

$$() \equiv x_1() + x_2(23) + x_3(12) + x_4(123) + x_5(132) + x_6(13)$$

Apply Loday's formula for the Koszul dual operad: include the signs, and observe that the association types are now reversed:

$$a \cdot bc \equiv x_1 ab \cdot c - x_2 ac \cdot b - x_3 ba \cdot c + x_4 ca \cdot b + x_5 bc \cdot a - x_6 cb \cdot a$$

Pass to the opposite algebra to "unreverse" the association types:

$$cb \cdot a \equiv x_1 c \cdot ba - x_2 b \cdot ca - x_3 c \cdot ab + x_4 b \cdot ac + x_5 a \cdot cb - x_6 a \cdot bc$$

Interchange a and c to get the identity permutation on the left:

$$ab \cdot c \equiv x_1 a \cdot bc - x_2 b \cdot ac - x_3 a \cdot cb + x_4 b \cdot ca + x_5 c \cdot ab - x_6 c \cdot ba$$

Put the permutations on the right in lexicographical order:

$$ab \cdot c \equiv x_1 a \cdot bc - x_3 a \cdot cb - x_2 b \cdot ac + x_4 b \cdot ca + x_5 c \cdot ab - x_6 c \cdot ba$$

We now see that the relation defining the original operad and the relation defining its (opposite) dual are related by this table:

i	1	2	3	4	5	6
σ_i	abc	acb	bac	bca	cab	cba
original operad	x_1	x_2	x_3	x_4	x_5	x_6
opposite dual	x_1	$-x_3$	$-x_2$	x_4	x_5	$-x_6$

Associativity is self-dual, as is the relation $(ab)c \equiv \lambda c(ab)$.
The Leibniz and Zinbiel relations form an opposite dual pair.

One-parameter families including Leibniz and Zinbiel

Consider a parameter $t \in F$, $t \neq \pm 1$, and Loday's parametrized relation with these coefficients:

$$x_1 = x_2 = \frac{t^2+t-1}{t^2-1}, \quad x_3 = x_4 = x_5 = \frac{1}{t^2-1}, \quad x_6 = \frac{t}{t^2-1}.$$

Taking the limit as $t \to \infty$ gives the Zinbiel identity:

$$x_1 = x_2 = 1, \qquad x_3 = x_4 = x_5 = x_6 = 0.$$

Of the 6 new relations discovered by our computational search, numbers 1, 3, 5 correspond to the values $t = 2$, $t = \frac{1}{2}$, $t = -\frac{1}{2}$:

$$(ab)c \equiv \tfrac{1}{3}[\,5\,a(bc) + 5\,a(cb) + b(ac) + b(ca) + c(ab) + 2\,c(ba)\,]$$

$$(ab)c \equiv \tfrac{1}{3}[\,a(bc) + a(cb) - 4\,b(ac) - 4\,b(ca) - 4\,c(ab) - 2\,c(ba)\,]$$

$$(ab)c \equiv \tfrac{1}{3}[\,5\,a(bc) + 5\,a(cb) - 4\,b(ac) - 4\,b(ca) - 4\,c(ab) + 2\,c(ba)\,]$$

The opposite dual given by Loday's formula is the following family:

$$x_1 = \frac{t^2+t-1}{t^2-1}, \quad x_2 = \frac{-1}{t^2-1}, \quad x_3 = \frac{-(t^2+t-1)}{t^2-1},$$
$$x_4 = x_5 = \frac{1}{t^2-1}, \quad x_6 = \frac{-t}{t^2-1}.$$

Taking the limit as $t \to \infty$ gives the Leibniz identity:

$$x_1 = 1, \qquad x_2 = 0, \qquad x_3 = -1, \qquad x_4 = x_5 = x_6 = 0.$$

Of the new 6 relations discovered by our computational search, numbers 2, 4, 6 correspond to the values $t = 2$, $t = \frac{1}{2}$, $t = -\frac{1}{2}$:

$$(ab)c \equiv \tfrac{1}{3}[\,5\,a(bc) - a(cb) - 5\,b(ac) + b(ca) + c(ab) - 2\,c(ba)\,]$$
$$(ab)c \equiv \tfrac{1}{3}[\,a(bc) + 4\,a(cb) - b(ac) - 4\,b(ca) - 4\,c(ab) + 2\,c(ba)\,]$$
$$(ab)c \equiv \tfrac{1}{3}[\,5\,a(bc) + 4\,a(cb) - 5\,b(ac) - 4\,b(ca) - 4\,c(ab) - 2\,c(ba)\,]$$

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