Enumeration of nilpotent loops of order $2q$, up to isotopy

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August 15, 2013

Statement of results

$q \geq 3$ a prime throughout.

Theorem (Daly-Vojtěchovský 09)

$$\mathcal{N}(2q) = \sum_{d \text{ divides } q-1} \frac{1}{d} \left(2^{(q-2)d} + \sum_{D \subset \mathrm{Pred}(d)} (-1)^{|D|} 2^{(q-2)\,\mathrm{gcd}(D)} \right)$$

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$$(x, a)(y, b) = (xy, a + b + \theta(x, y))$$

  for some normalized cocycle θ
- If order $2q$, take $F = \mathbb{Z}/q$, $A = \mathbb{Z}/2$.

Action of Map_0

For all $\tau \in \mathrm{Map}(F, A)$ s.t. $\tau(1) = 0$, form

$$\widehat{\tau}(x, y) = \tau(xy) - \tau(x) - \tau(y)$$

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Fact 1:
$Q_\theta \simeq Q_{\theta+\widehat{\tau}}$ via $(x, a) \mapsto (x, a + \tau(x))$.

Action of $\mathrm{Aut}(F, A)$

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For all $\alpha \in \mathrm{Aut}(F)$, $h \in \mathrm{Aut}(A)$, consider

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Then $Q_\theta \simeq Q_{h\,\theta(\alpha^{-1},\alpha^{-1})}$

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Similar action of $\mathrm{Atp}(F) \times \mathrm{Aut}(A)$?

Action of Atp(F, A)

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Fact 2: For $\alpha \in \mathrm{Aut}(F)$, $h \in \mathrm{Aut}(A)$, $Q_\theta \simeq Q_{h\,\theta(\alpha^{-1},\alpha^{-1})}$

For $t = (\alpha, \beta, \gamma) \in \mathrm{Atp}(F)$, $h \in \mathrm{Aut}(A)$,

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Compose with the normalizer map N.
This defines the desired action. It is compatible with the previous ones.
Moreover, θ, μ are in the same orbit $\Rightarrow Q_\theta \approx Q_\mu$.

Separability

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Theorem
If $\mathrm{Aut}(Q_\theta)$ *acts transitively on*

$$\{K \le Z(Q_\theta);\ K \cong A,\ Q_\theta/K \simeq F\}$$

then θ *is isotopy separable.*

Separability

Thank you