Q loop, $a \in Q$

$x \in Q$ $\quad L_a(x) = ax$ left translation

$\qquad R_a(x) = xa$ right "

$$\mathrm{Mlt}\, Q = \langle L_a, R_a \mid a \in Q \rangle$$

$$N_\lambda = N_\lambda(Q) = \{ a \in Q \mid a(xy) = (ax)y \ \forall x, y \in Q \}$$

$$N_\mu = N_\mu(Q) = \{ a \in Q \mid (xa)y = x(ay) \ \forall x, y \in Q \}$$

$$N_\rho = N_\rho(Q) = \{ a \in Q \mid (xy)a = x(ya) \ \forall x, y \in Q \}$$

$$N = N(Q) = N_\lambda \cap N_\mu \cap N_\rho$$

nucleus of Q

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center of Q:

$$\underline{Z(Q) = \{} a \in N(Q) \mid ax = xa \ \forall x \in Q \}$$

Prop. $Z(\mathrm{Mlt}\, Q) = \{ L_a \mid a \in Z(Q) \}$

Def: Q centrally nilpotent of class n, if

$Z_0 = 1$, $Z_1 = Z(Q), \ldots, Z_i / Z_{i-1} \cong Z(Q/Z_{i-1})$

$Z_{n-1} \neq Q$, $Z_n = Q$

$C(Q)$ commutant of Q:

$C(Q) = \{ x \in Q \mid L_x = R_x \}$

$A(Q)$: the associator subloop of Q the least normal subloop s.t. $Q/A(Q)$ group

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MOUFANG LOOPS

1. $((xy)x)z = x(y(xz))$
2. $((xy)z)y = x(y(zy)) \quad \forall x, y, z \in Q$
3. $(xy)(zx) = x(yz)x$

$A = \{L_x \mid x \in Q\}, \quad B = \{R_x \mid x \in Q\}$

i) $L_x L_y L_x \in A, \quad R_x R_y R_x \in B \quad \forall x, y \in Q$

ii) $L_x (L_y)^{(R_x)^{-1}} \in A$

$R_x R_y^{(L_x)^{-1}} \in B \quad \forall x, y \in Q$

$G = \mathrm{Mlt}\, Q$

$N = N_\lambda = N_\mu = N_\rho$ normal subloop of Q

$C_G(A) = \{R_x \mid x \in N\}$

$C_G(B) = \{L_x \mid x \in N\}$

PHILLIPS' PROBLEM:

whether there exists a Moufang loop of odd order with trivial nucleus?

G. Glauberman, Doro

Q Moufang loop of odd order

$\Downarrow$

1. Q is solvable
2. Mlt Q is solvable
3. |Mlt Q| has the same prime divisors as |Q|

$$\boxed{S_3 \leq \mathrm{Aut}\,\mathrm{Mlt}\,Q}$$

$\Uparrow$

Q with trivial nucleus

Theorem /P. Cs./

Q Moufang loop of odd order
$C(Q)$ is not trivial
$\Downarrow$
Q has nontrivial nucleus

Theorem /S. Gagola/:

Q Moufang loop
$\Downarrow$
$C(Q)$ normal subloop of Q

theory of connected transversals:

$\mathrm{Mlt}\,Q = \langle L_a, R_a \mid a \in Q \rangle$

$\mathrm{Inn}\,Q = \mathrm{stab}(1)$

$A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$

1. A and B are left transversals to $\mathrm{Inn}\,Q$
2. $\langle A, B \rangle = \mathrm{Mlt}\,Q$
3. $[A, B] \leq \mathrm{Inn}\,Q$
4. $\mathrm{core}_{\mathrm{Mlt}\,Q}\,\mathrm{Inn}\,Q = 1$

Theorem /M. Niemenmaa, T. Kepka/
G group
Then $\exists Q$ loop : $G \cong \mathrm{Mlt}\,Q$

$\Updownarrow$

1. $\exists H \leq G$, $\exists A, B$ left transv. to H
2. $\langle A, B \rangle = G$
3. $[A, B] \leq H$ — A, B H-connected transv.
4. $\mathrm{core}_G H = 1$

Theorem /P. Cs./

Q finite Moufang loop

N(Q) the nucleus of Q

$$Z(Q/N(Q)) \neq 1$$
$$\Downarrow$$
$$Z(Q) \neq 1$$

Theorem /P. Cs./

Q Moufang loop of odd order

$$C(Q) \neq 1$$
$$\Downarrow$$
$$Z(Q) \neq 1$$

Proof. Q minimal counterexample

$G = \mathrm{Mlt}\,Q$, $A = \{L_x \mid x \in Q\}$, $B = \{R_x \mid x \in Q\}$

$\exists\, 1 \neq x_0 \in C(Q) \qquad L_{x_0} = R_{x_0}$

$N(Q) \neq 1$

1. $x_0 \in N(Q) \Rightarrow L_{x_0} \in C_G(A) \cap C_G(B)$
$$\Downarrow$$
$$L_{x_0} \in Z(G) \Rightarrow x_0 \in Z(Q) \quad \lightning$$

2. $x_0 \notin N(Q)$

$$|Q/N(Q)| \lneq |Q|$$
$$\Downarrow$$
$$Z(Q/N(Q)) \neq 1 \Rightarrow Z(Q) \neq 1 \quad \lightning$$

Definition : Q Moufang loop

$C(Q)$ the commutant of Q

Q commutantly nilpotent:

$C_0 = 1$, $C_1 = C(Q)$, .. $C_i/C_{i-1} = C(Q/C_{i-1})$

$C_{n-1} \neq Q$, $C_n = Q$

Theorem /P. G./

Q Moufang loop of odd order

Then Q centrally nilpotent

$\Updownarrow$

Q commutantly nilpotent

Corollary / OLD RESULT/:

Q commutative Moufang loop of odd order

$\Downarrow$

Q centrally nilpotent

Theorem /P. G./

Q finite Moufang loop

Then Q centrally nilpotent

$\Updownarrow$

1. $Q/N(Q)$ centrally nilpotent
2. $Q/A(Q)$ centrally nilpotent

Definition: Q Moufang loop

Q nuclearly nilpotent, if

$N_0 = 1$, $N_1 = N(Q)$

$N_i / N_{i-1} = N(Q/N_{i-1})$

$N_{k-1} \neq Q$, $N_k = Q$

Theorem /P. G./

Q finite Moufang loop

Then Q centrally nilpotent

$\Updownarrow$

1. Q nuclearly nilpotent
2. $Q/A(Q)$ centrally nilpotent

Corollary /P. Cs./

Q Moufang loop of odd order

Then Q commutantly nilpotent

$\Updownarrow$

1. Q nuclearly nilpotent
2. $Q/\mathcal{A}(Q)$ centrally nilpotent