Fine gradings and gradings by root systems on simple Lie algebras

Alberto Elduque

Universidad de Zaragoza

3rd Mile High Conference on Nonassociative Mathematics

August 2013

Gradings

Gradings by root systems

Fine gradings and gradings by root systems

Gradings

Gradings by root systems

Fine gradings and gradings by root systems

Gradings

$\mathbb{F}$ algebraically closed field of characteristic 0.

G abelian group, $\mathcal{A}$ algebra over $\mathbb{F}$.
(Only finite-dimensional algebras will be considered here.)

Gradings

$\mathbb{F}$ algebraically closed field of characteristic 0.

G abelian group, $\mathcal{A}$ algebra over $\mathbb{F}$.
(Only finite-dimensional algebras will be considered here.)

G-grading on $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{g+h} \qquad \forall g, h \in G.$$

Universal group

Universal group

- Support:

$$\text{Supp}\ \Gamma := \{g \in G : \mathcal{A}_g \neq 0\}.$$

Universal group

- Support:

$$\mathrm{Supp}\ \Gamma := \{g \in G : \mathcal{A}_g \neq 0\}.$$

- Universal group: This is the group $(U(\Gamma), \boxplus)$ generated by $\mathrm{Supp}\ \Gamma$ subject to the relations $g \boxplus h = g + h$ for any $g, h \in \mathrm{Supp}\ \Gamma$ such that $g + h \in \mathrm{Supp}\ \Gamma$:

$$U(\Gamma) := \langle \mathrm{Supp}\ \Gamma \rangle / \langle g \boxplus h \boxplus \big(-(g+h)\big) : g, h, g+h \in \mathrm{Supp}\ \Gamma \rangle.$$

Universal group

- Support:
$$\mathrm{Supp}\ \Gamma := \{g \in G : \mathcal{A}_g \neq 0\}.$$

- Universal group: This is the group $(U(\Gamma), \boxplus)$ generated by $\mathrm{Supp}\ \Gamma$ subject to the relations $g \boxplus h = g + h$ for any $g, h \in \mathrm{Supp}\ \Gamma$ such that $g + h \in \mathrm{Supp}\ \Gamma$:
$$U(\Gamma) := \langle \mathrm{Supp}\ \Gamma \rangle / \langle g \boxplus h \boxplus \big(-(g+h)\big) : g, h, g+h \in \mathrm{Supp}\ \Gamma \rangle.$$

Universal group

- Support:
$$\mathrm{Supp}\,\Gamma := \{g \in G : \mathcal{A}_g \neq 0\}.$$

- Universal group: This is the group $(U(\Gamma), \boxplus)$ generated by $\mathrm{Supp}\,\Gamma$ subject to the relations $g \boxplus h = g + h$ for any $g, h \in \mathrm{Supp}\,\Gamma$ such that $g + h \in \mathrm{Supp}\,\Gamma$:
$$U(\Gamma) := \langle \mathrm{Supp}\,\Gamma \rangle / \langle g \boxplus h \boxplus \bigl(-(g+h)\bigr) : g, h, g+h \in \mathrm{Supp}\,\Gamma \rangle.$$
Γ can then be realized as a grading by $U(\Gamma)$.

Example

$$\mathcal{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathrm{span}\left\{ e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),\ h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \right\}.$$

Example

$$\mathcal{L} = \mathfrak{sl}_2(\mathbb{F}) = \mathrm{span}\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

$$\Gamma : \mathcal{L} = \underset{\bar{1}}{\underset{\uparrow}{\mathbb{F}e}} \oplus \underset{\bar{0}}{\underset{\uparrow}{\mathbb{F}h}} \oplus \underset{\bar{2}}{\underset{\uparrow}{\mathbb{F}f}} \qquad \mathbb{Z}_3\text{-grading}$$

Example

$$\mathcal{L} = \mathfrak{sl}_2(\mathbb{F}) = \text{span}\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

$$\begin{array}{ccccccccc} \Gamma : \mathcal{L} & = & \mathbb{F}e & \oplus & \mathbb{F}h & \oplus & \mathbb{F}f & \qquad & \mathbb{Z}_3\text{-grading} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \bar{1} & & \bar{0} & & \bar{2} & & \end{array}$$

$U(\Gamma) = \mathbb{Z}$, because $[e, e] = [f, f] = 0$.

Fine gradings

$$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g'\in G'} \mathcal{A}'_{g'}, \quad \text{gradings on } \mathcal{A}.$$

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g'\in G'} \mathcal{A}'_{g'}, \quad$ gradings on $\mathcal{A}$.

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad$ gradings on $\mathcal{A}$.

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.

  Then Γ' is a *coarsening* of Γ.

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad$ gradings on $\mathcal{A}$.

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.

  Then Γ' is a *coarsening* of Γ.

- Γ is *fine* if it admits no proper refinement.

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad$ gradings on $\mathcal{A}$.

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a *coarsening* of Γ.

- Γ is *fine* if it admits no proper refinement.

Remark

Any grading is a coarsening of a fine grading.

Example: Cartan grading

Example: Cartan grading

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

(root space decomposition of a semisimple complex Lie algebra).

Example: Cartan grading

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

(root space decomposition of a semisimple complex Lie algebra).

This is a fine grading by $\mathbb{Z}\Phi \simeq \mathbb{Z}^n$, $n = \operatorname{rank} \mathfrak{g}$.

Example: Pauli matrices

Example: Pauli matrices

$\mathcal{A} = \mathsf{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

$\mathcal{A}$ becomes a graded division algebra.

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar\imath,\bar\jmath)\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathcal{A}_{(\bar\imath,\bar\jmath)}, \qquad \mathcal{A}_{(\bar\imath,\bar\jmath)} = \mathbb{F}X^iY^j.$$

$\mathcal{A}$ becomes a graded division algebra.

This grading induces a fine grading on $\mathfrak{sl}_n(\mathbb{F})$:

$$\mathfrak{sl}_n(\mathbb{F}) = \bigoplus_{0\neq(\bar\imath,\bar\jmath)\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathbb{F}X^iY^j.$$

Example: Octonions

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

- $\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\mathbf{i}$, $\qquad$ $\mathbb{Z}_2$-graded;

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

- $\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\mathbf{i}$, $\mathbb{Z}_2$-graded;
- $\mathbb{H} = \mathbb{K} \oplus \mathbb{K}\mathbf{j}$, $\mathbb{Z}_2^2$-graded;

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

- $\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\mathbf{i}$, $\qquad \mathbb{Z}_2$-graded;
- $\mathbb{H} = \mathbb{K} \oplus \mathbb{K}\mathbf{j}$, $\qquad \mathbb{Z}_2^2$-graded;
- $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{l}$, $\qquad \mathbb{Z}_2^3$-graded.

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

- $\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\mathbf{i}$, $\mathbb{Z}_2$-graded;
- $\mathbb{H} = \mathbb{K} \oplus \mathbb{K}\mathbf{j}$, $\mathbb{Z}_2^2$-graded;
- $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{l}$, $\mathbb{Z}_2^3$-graded.

Example: Octonions

The Cayley algebra $\mathbb{O}$ is obtained from the ground field $\mathbb{F}$ by means of the Cayley-Dickson doubling process:

- $\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\mathbf{i}$, $\mathbb{Z}_2$-graded;
- $\mathbb{H} = \mathbb{K} \oplus \mathbb{K}\mathbf{j}$, $\mathbb{Z}_2^2$-graded;
- $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{l}$, $\mathbb{Z}_2^3$-graded.

All these are fine gradings.

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \text{ for all } H \in \mathfrak{h}\}$; and

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \text{ for all } H \in \mathfrak{h}\}$; and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \text{ for all } H \in \mathfrak{h}\}$; and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

Gradings by root systems

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ over $\mathbb{F}$ is *graded by the reduced root system* Φ, or Φ-graded, if:

1. $\mathcal{L}$ contains as a subalgebra a finite-dimensional semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right)$ whose root system is Φ relative to a Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \text{ for all } H \in \mathfrak{h}\}$; and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

The subalgebra $\mathfrak{g}$ is said to be a *grading subalgebra* of $\mathcal{L}$.

Gradings by root systems

For irreducible Φ, view $\mathcal{L}$ as a module for $\mathfrak{g}$. As such it is a direct sum of copies of the adjoint, the little adjoint and the trivial modules. We may collect isomorphic irreducible $\mathfrak{g}$-submodules in $\mathcal{L}$:

$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathcal{W} \otimes \mathcal{B}) \oplus \mathcal{D},$$

Gradings by root systems

For irreducible Φ, view $\mathcal{L}$ as a module for $\mathfrak{g}$. As such it is a direct sum of copies of the adjoint, the little adjoint and the trivial modules. We may collect isomorphic irreducible $\mathfrak{g}$-submodules in $\mathcal{L}$:

$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathcal{W} \otimes \mathcal{B}) \oplus \mathcal{D},$$

where

- the grading subalgebra $\mathfrak{g}$ is identified with $\mathfrak{g} \otimes 1$ for a distinguished element $1 \in \mathcal{A}$,
- $\mathcal{W}$ is 0 if Φ is simply laced, while $\mathcal{W}$ is the little adjoint module (the irreducible $\mathfrak{g}$-module whose highest weight is the highest short root) otherwise,
- $\mathcal{D}$ is the centralizer of $\mathfrak{g} \simeq \mathfrak{g} \otimes 1$, and hence it is a subalgebra of $\mathcal{L}$.

Coordinate algebra

Coordinate algebra

The Lie bracket in $\mathcal{L}$ induces a multiplication on the sum

$$\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}.$$

$\mathfrak{a}$ becomes a unital nonassociative algebra: the coordinate algebra.

Coordinate algebra

The Lie bracket in $\mathcal{L}$ induces a multiplication on the sum

$$\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}.$$

$\mathfrak{a}$ becomes a unital nonassociative algebra: the coordinate algebra.

Associative, alternative, Jordan and structurable algebras appear as coordinate algebras.

Coordinate algebra

The Lie bracket in $\mathcal{L}$ induces a multiplication on the sum

$$\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}.$$

$\mathfrak{a}$ becomes a unital nonassociative algebra: the coordinate algebra.

Associative, alternative, Jordan and structurable algebras appear as coordinate algebras.

The elements of the Lie subalgebra $\mathcal{D}$ act as derivations on $\mathfrak{a}$.

Example

Example

Let $\mathbb{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Example

Let $\mathbb{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Consider Tits' construction:

$$\mathcal{T}(\mathbb{O}, \mathcal{J}) = \mathfrak{der}\,\mathbb{O} \oplus (\mathbb{O}_0 \otimes \mathcal{J}_0) \oplus \mathfrak{der}\,\mathcal{J}.$$

Here $\mathfrak{g} = \mathfrak{der}\,\mathbb{O}$ is the simple Lie algebra of type G_2, $\mathcal{W} = \mathbb{O}_0$ is its little adjoint module.

Example

Let $\mathbb{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Consider Tits' construction:

$$\mathcal{T}(\mathbb{O}, \mathcal{J}) = \mathfrak{der}\,\mathbb{O} \oplus (\mathbb{O}_0 \otimes \mathcal{J}_0) \oplus \mathfrak{der}\,\mathcal{J}.$$

Here $\mathfrak{g} = \mathfrak{der}\,\mathbb{O}$ is the simple Lie algebra of type G_2, $\mathcal{W} = \mathbb{O}_0$ is its little adjoint module.

$\mathcal{T}(\mathbb{O}, \mathcal{J})$ is graded by the root system G_2,
with coordinate algebra $\mathcal{J}$.

Example

Let $\mathbb{O}$ be the Cayley algebra over $\mathbb{F}$, and let $\mathcal{J}$ be a central simple degree 3 Jordan algebra.

Consider Tits' construction:

$$\mathcal{T}(\mathbb{O}, \mathcal{J}) = \mathfrak{der}\, \mathbb{O} \oplus (\mathbb{O}_0 \otimes \mathcal{J}_0) \oplus \mathfrak{der}\, \mathcal{J}.$$

Here $\mathfrak{g} = \mathfrak{der}\, \mathbb{O}$ is the simple Lie algebra of type G_2, $\mathcal{W} = \mathbb{O}_0$ is its little adjoint module.

$\mathcal{T}(\mathbb{O}, \mathcal{J})$ is graded by the root system G_2,
with coordinate algebra $\mathcal{J}$.

Remark

An extension of Tits construction gives, up to isomorphisms, all G_2-graded Lie algebras (Benkart-Zelmanov).

Nonreduced root systems

Nonreduced root systems

Berman-Moody's definition can be extended to cover nonreduced root systems, thus considering, in the irreducible case, BC_r-graded Lie algebras (Benkart-Smirnov, Allison-Benkart-Gao).

Nonreduced root systems

Berman-Moody's definition can be extended to cover nonreduced root systems, thus considering, in the irreducible case, BC_r-graded Lie algebras (Benkart-Smirnov, Allison-Benkart-Gao).

An extra summand appears in the decomposition into isotypical components:

$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathcal{W} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D},$$

Nonreduced root systems

Berman-Moody's definition can be extended to cover nonreduced root systems, thus considering, in the irreducible case, BC_r-graded Lie algebras (Benkart-Smirnov, Allison-Benkart-Gao).

An extra summand appears in the decomposition into isotypical components:

$$\mathcal{L} = (\mathfrak{g} \otimes \mathcal{A}) \oplus (\mathcal{W} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D},$$

The coordinate algebra is then $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$.

Gradings

Gradings by root systems

Fine gradings and gradings by root systems

Some properties of fine gradings

Proposition

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G. Then:

Some properties of fine gradings

Proposition

Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ *be a fine grading on the simple Lie algebra* $\mathcal{L}$ *with universal group* G*. Then:*

- *The neutral homogeneous component* $\mathcal{L}_0$ *is a toral subalgebra of* $\mathcal{L}$ *(i.e.,* $\operatorname{ad} \mathcal{L}_0$ *consists of commuting diagonalizable operators in* $\mathcal{L}$*).*

Some properties of fine gradings

Proposition

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ *be a fine grading on the simple Lie algebra* $\mathcal{L}$ *with universal group* G*. Then:*

- *The neutral homogeneous component* $\mathcal{L}_0$ *is a toral subalgebra of* $\mathcal{L}$ *(i.e.,* $\operatorname{ad} \mathcal{L}_0$ *consists of commuting diagonalizable operators in* $\mathcal{L}$*).*
- *The dimension of* $\mathcal{L}_0$ *coincides with the free rank of* G*.*

Some properties of fine gradings

Proposition (continued)

Some properties of fine gradings

Proposition (continued)

- *Let* $\operatorname{tor}(G)$ *be the torsion subgroup of* G*. The coarsening*

$$\bar{\Gamma} : \mathcal{L} = \bigoplus_{\bar{g} \in G/\operatorname{tor}(G)} \bar{\mathcal{L}}_{\bar{g}},$$

where $\bar{\mathcal{L}}_{\bar{g}} = \bigoplus_{h \in \operatorname{tor}(G)} \mathcal{L}_{g+h}$*, is the weight space decomposition of* $\mathcal{L}$ *relative to* $\mathcal{L}_0$*.*

Some properties of fine gradings

Proposition (continued)

- *Let* tor(G) *be the torsion subgroup of* G. *The coarsening*

$$\bar{\Gamma} : \mathcal{L} = \bigoplus_{\bar{g}\in G/\operatorname{tor}(G)} \bar{\mathcal{L}}_{\bar{g}},$$

where $\bar{\mathcal{L}}_{\bar{g}} = \bigoplus_{h\in\operatorname{tor}(G)} \mathcal{L}_{g+h}$, *is the weight space decomposition of* $\mathcal{L}$ *relative to* $\mathcal{L}_0$.

Some properties of fine gradings

Proposition (continued)

- *Let* tor(G) *be the torsion subgroup of* G. *The coarsening*

$$\bar{\Gamma} : \mathcal{L} = \bigoplus_{\bar{g} \in G/\operatorname{tor}(G)} \bar{\mathcal{L}}_{\bar{g}},$$

where $\bar{\mathcal{L}}_{\bar{g}} = \bigoplus_{h \in \operatorname{tor}(G)} \mathcal{L}_{g+h}$, *is the weight space decomposition of* $\mathcal{L}$ *relative to* $\mathcal{L}_0$.

That is, for any $\bar{g} \in \operatorname{Supp} \bar{\Gamma}$, *there is a linear form* $\alpha \in \mathcal{L}_0^*$ *such that* $\mathcal{L}_{\bar{g}}$ *equals*

$$\mathcal{L}(\alpha) = \{x \in \mathcal{L} : [h, x] = \alpha(h)x \ \forall h \in \mathcal{L}_0\}.$$

Root system

Moreover,

Root system

Moreover,

Proposition

- *The set*

$$\Phi = \{\alpha \in \mathcal{L}_0^* \setminus \{0\} : \mathcal{L}(\alpha) \neq 0\}$$

is a (possibly nonreduced) irreducible root system.

Root system

Moreover,

Proposition

- *The set*

$$\Phi = \{\alpha \in \mathcal{L}_0^* \setminus \{0\} : \mathcal{L}(\alpha) \neq 0\}$$

is a (possibly nonreduced) irreducible root system.

- *The map*

$$\begin{aligned} \pi : G &\longrightarrow \mathbb{Z}\Phi \\ g &\mapsto \ \alpha \quad \textit{such that } \mathcal{L}_g \subseteq \mathcal{L}(\alpha), \end{aligned}$$

is a surjective group homomorphism, with $\ker \pi = \mathrm{tor}(G)$.

Fine gradings and gradings by root systems

Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Let Φ be the associated root system.

Let $\tilde{G}$ be a complement of $\text{tor}(G)$: $\quad G = \tilde{G} \oplus \text{tor}(G)$, $\quad$ and consider the subalgebra

$$\mathfrak{g} = \bigoplus_{g\in\tilde{G}} \mathcal{L}_g .$$

Fine gradings and gradings by root systems

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Let Φ be the associated root system.

Let $\tilde{G}$ be a complement of $\mathrm{tor}(G)$: $\quad G = \tilde{G} \oplus \mathrm{tor}(G)$, $\quad$ and consider the subalgebra

$$\mathfrak{g} = \bigoplus_{g \in \tilde{G}} \mathcal{L}_g .$$

Theorem

$\mathcal{L}$ is graded by the root system Φ with grading subalgebra $\mathfrak{g}$.

Grading on the coordinate algebra

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Grading on the coordinate algebra

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Then Γ induces:

Grading on the coordinate algebra

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Then Γ induces:

(i) a grading by the irreducible root system Φ,

Grading on the coordinate algebra

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a fine grading on the simple Lie algebra $\mathcal{L}$ with universal group G.

Then Γ induces:

(i) a grading by the irreducible root system Φ,

(ii) a fine grading by $\mathrm{tor}(G)$ on the coordinate algebra $\mathfrak{a}$, which satisfies $\mathfrak{a}_0 = \mathbb{F}1$.

Examples

The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is ≥ 3 are the following:

Examples

The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is ≥ 3 are the following:

- The Cartan gradings on F_4, E_6, E_7 and E_8.

Examples

The fine gradings on the exceptional simple Lie algebras such that the free rank of its universal group is ≥ 3 are the following:

- The Cartan gradings on F_4, E_6, E_7 and E_8.
- A fine grading on E_7 by $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ related to a grading by the root system C_3:

  $$\mathfrak{e}_7 = \mathcal{T}(\mathbb{O}, \mathcal{H}_3(\mathbb{H})) = \mathfrak{der}\,\mathbb{O} \oplus (\mathbb{O}_0 \otimes \mathcal{H}_3(\mathbb{H})_0) \oplus \mathfrak{der}\,\mathcal{H}_3(\mathbb{H}).$$

  Here $\mathfrak{der}\,\mathcal{H}_3(\mathbb{H})$ is the simple Lie algebra of type C_3, and $\mathcal{H}_3(\mathbb{H})_0$ is its little adjoint module.
  The coordinate algebra is $\mathbb{O}$, endowed with its $\mathbb{Z}_2^3$-grading.

Examples

- Gradings by $\mathbb{Z}^4 \times \mathbb{Z}_2^{r-5}$ on E_r $(r = 6, 7, 8)$ related to gradings by the root system F_4:

$$\mathfrak{e}_r = \mathcal{T}(\mathcal{C}, \mathcal{H}_3(\mathbb{O})) = \mathfrak{der}\,\mathcal{C} \oplus (\mathcal{C}_0 \otimes \mathcal{H}_3(\mathbb{O})_0) \oplus \mathfrak{der}\,\mathcal{H}_3(\mathbb{O}).$$

Here $\mathfrak{der}\,\mathcal{H}_3(\mathbb{O})$ is the simple Lie algebra of type F_4, and $\mathcal{H}_3(\mathbb{O})_0$ is its little adjoint module.
The coordinate algebra is $\mathcal{C} = \mathbb{K}$, $\mathbb{H}$ or $\mathbb{O}$ endowed, respectively, with its fine grading by $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$.

Classification of fine gradings

Classification of fine gradings

- The fine gradings on simple Lie algebras with infinite universal groups are thus obtained by combining a grading by a root system and a 'special grading' on the coordinate algebra.

Classification of fine gradings

- The fine gradings on simple Lie algebras with infinite universal groups are thus obtained by combining a grading by a root system and a 'special grading' on the coordinate algebra.

- The classification of the fine gradings on the classical simple Lie algebras was completed in 2010,

  on G_2 in 2006 (Draper–Martín-González, and independently Bahturin–Tvalavadze),

  on F_4 in 2009 (Draper–Martín-González), and

  on E_6 in 2012 (Draper–Viruel, preprint).

Classification of fine gradings

- The fine gradings on simple Lie algebras with infinite universal groups are thus obtained by combining a grading by a root system and a 'special grading' on the coordinate algebra.

- The classification of the fine gradings on the classical simple Lie algebras was completed in 2010,

  on G_2 in 2006 (Draper–Martín-González, and independently Bahturin–Tvalavadze),

  on F_4 in 2009 (Draper–Martín-González), and

  on E_6 in 2012 (Draper–Viruel, preprint).

- A whole bunch of fine gradings has been obtained, using the relationship of fine gradings and gradings by root systems, for the exceptional simple Lie algebras E_7 and E_8, but the classification of the fine gradings for these Lie algebras is not yet complete.

That's all.
Thanks