A class of latin squares derived from finite abelian groups

Anthony B. Evans

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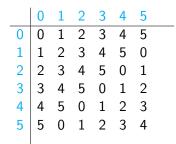
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Abstract

We consider latin squares obtained by extending the Cayley tables of finite abelian groups, and give preliminary results on the existence/nonexistence of latin squares orthogonal to these.

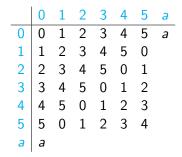
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The Cayley table of $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$



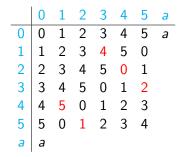
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Extend the symbol set to $\{0, 1, 2, 3, 4, 5, a\}$.



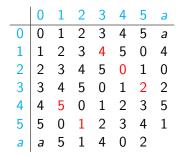
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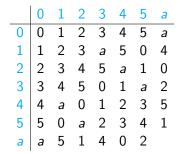
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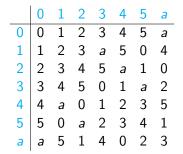
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 $G = \{g_0, \ldots, g_{m-1}\}, g_0 = 0$, an abelian group.

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Form a latin square, $Ext_{\theta}(G; a)$.

	g0	g ₁		g_{m-1}	а	
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$Ext_{\theta}(G; a) =$	÷		Α		В	
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The $g_i g_j$ th entry of A is $\begin{cases} a & \text{if } \theta(g_i) = g_j, \\ g_i + g_j & \text{if } \theta(g_i) \neq g_j. \end{cases}$

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Characterizing θ and w.

Define η by $\eta(g_i) = \theta(g_i) + g_i$.

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Lemma

• If G has a unique involution δ , then

 $w = \delta$, and

 $\theta \text{ is a near complete mapping of } G, \text{ i.e., } \eta \text{ is a bijection} \\ \{g_1, \dots, g_{m-1}\} \rightarrow \{g_0, \dots, g_{m-1}\} \setminus \{\delta\}.$

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- $\begin{array}{l} \theta \text{ is a near complete mapping of } G, \text{ i.e., } \eta \text{ is a bijection} \\ \{g_1, \dots, g_{m-1}\} \rightarrow \{g_0, \dots, g_{m-1}\} \setminus \{\delta\}. \end{array}$
- Otherwise

w = 0, and

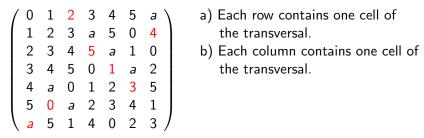
 θ is a "complete mapping" of G, i.e., η is a bijection $\{g_1, \ldots, g_{m-1}\} \rightarrow \{g_1, \ldots, g_{m-1}\}.$

A transversal in $Ext(\mathbb{Z}_6; a)$

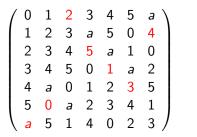


a) Each row contains one cell of the transversal.

A transversal in $Ext(\mathbb{Z}_6; a)$



A transversal in $Ext(\mathbb{Z}_6; a)$



- a) Each row contains one cell of the transversal.
- b) Each column contains one cell of the transversal.
- c) Each symbol appears exactly once in the transversal.

Deviations and the Δ - lemma.

Let *L* be a latin square with rows and columns indexed by the elements $\{g_0, \ldots, g_{m-1}\}$ of an abelian group *G*.

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If cell C is in row g_i and column g_i , and its entry is g_k , then

$$dev(C) = g_k - (g_i + g_j).$$

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$$dev(C) = g_k - (g_i + g_j).$$

The Δ -lemma

Let C_1, \ldots, C_m be the cells of a transversal of L.

• If G has a unique involution δ , then

$$\sum_{i=1}^m dev(C_i) = \delta.$$

Otherwise

$$\sum_{i=1}^m dev(C_i) = 0.$$

Orthogonal latin squares.

A pair of orthogonal latin squares of order 5.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 2 & 4 & 1 & 3 \\ 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \end{pmatrix}$$

$$\searrow \text{superimposed} \checkmark$$

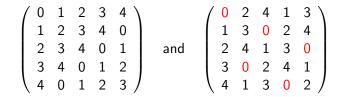
$$\begin{pmatrix} 0, 0 & 1, 2 & 2, 4 & 3, 1 & 4, 3 \\ 1, 1 & 2, 3 & 3, 0 & 4, 2 & 0, 4 \\ 2, 2 & 3, 4 & 4, 1 & 0, 3 & 1, 0 \\ 3, 3 & 4, 0 & 0, 2 & 1, 4 & 2, 1 \\ 4, 4 & 0, 1 & 1, 3 & 2, 0 & 3, 2 \end{pmatrix}$$

Each ordered pair of symbols appears exactly once.

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Orthogonality and transversals.

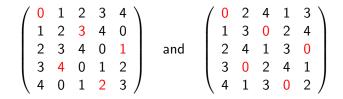
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The red entries in the second square are all 0.

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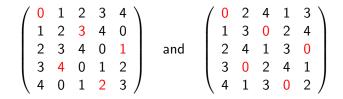
The red entries in the second square are all 0.

The corresponding entries in the first square form a transversal.

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Orthogonality and transversals.

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Lemma

A latin square has an orthogonal mate if and only its set of cells can be partitioned by some set of transversals.

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Definition

A bachelor square is a latin square without an orthogonal mate.

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Theorem

If G does not have a unique involution, then Ext(G; a) is a confirmed bachelor square.

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Examples

If m ≡ 0 (mod 4) and G = Z₂ × Z_{m/2}, then Ext(G; a) is a confirmed bachelor square of order m + 1.

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- If m ≡ 0 (mod 4) and G = Z₂ × Z_{m/2}, then Ext(G; a) is a confirmed bachelor square of order m + 1.
- If m is odd and $G = \mathbb{Z}_m$, then Ext(G; a) is a confirmed bachelor square of order m + 1.

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$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & a \\ 1 & a & 3 & 4 & 5 & 6 & 0 & 2 \\ 2 & 3 & a & 5 & 6 & 0 & 1 & 4 \\ 3 & 4 & 5 & a & 0 & 1 & 2 & 6 \\ 4 & 5 & 6 & 0 & a & 2 & 3 & 1 \\ 5 & 6 & 0 & 1 & 2 & a & 4 & 3 \\ 6 & 0 & 1 & 2 & 3 & 4 & a & 5 \\ a & 2 & 4 & 6 & 1 & 3 & 5 & 0 \end{pmatrix}$$

Suppose there is a transversal through the "a" in the last row.

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Suppose there is a transversal through the "a" in the last row. None of the red entries can be on this transversal.

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$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & 3 & 4 & 5 & 6 & 0 & 2 \\ & 3 & 5 & 6 & 0 & 1 & 4 \\ & 4 & 5 & 0 & 1 & 2 & 6 \\ & 5 & 6 & 0 & 2 & 3 & 1 \\ & 6 & 0 & 1 & 2 & 4 & 3 \\ & 0 & 1 & 2 & 3 & 4 & 5 \\ a & & & & & & \end{pmatrix}$$

Suppose there is a transversal through the "a" in the last row. None of the red entries can be on this transversal. Remove these.

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$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & 3 & 4 & 5 & 6 & 0 & 2 \\ & 3 & 5 & 6 & 0 & 1 & 4 \\ & 4 & 5 & 0 & 1 & 2 & 6 \\ & 5 & 6 & 0 & 2 & 3 & 1 \\ & 6 & 0 & 1 & 2 & 4 & 3 \\ & 0 & 1 & 2 & 3 & 4 & 5 \\ a & & & & & & \end{pmatrix}$$

Suppose there is a transversal through the "*a*" in the last row. None of the red entries can be on this transversal. Remove these. Rearrange columns: move "red" column.

A B M A B M

$$Ext(\mathbb{Z}_7; a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 0 \\ 4 & 3 & 5 & 6 & 0 & 1 \\ 6 & 4 & 5 & 0 & 1 & 2 \\ 1 & 5 & 6 & 0 & 2 & 3 \\ 3 & 6 & 0 & 1 & 2 & 4 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ a & & & & & \end{pmatrix}$$

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Suppose there is a transversal through the "*a*" in the last row. None of the red entries can be on this transversal. Remove these.

Rearrange columns: move "red" column.

Compute deviations.

The transversal must contain exactly one cell from each column and the deviations must add to 0.

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None of the red entries can be on this transversal. Remove these. Rearrange columns: move "red" column.

Compute deviations.

The transversal must contain exactly one cell from each column and the deviations must add to 0.

This is impossible: a contradiction.

Some bachelor/monogamous squares?

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A monogamous square is a latin square that has an orthogonal mate, but is not contained in a set of three pairwise orthogonal latin squares.

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A monogamous square is a latin square that has an orthogonal mate, but is not contained in a set of three pairwise orthogonal latin squares.

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If G has a unique involution, then $E \times t_{\theta}(G; a)$ is a either a bachelor square or a monogamous square.

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If G has a unique involution, then $E \times t_{\theta}(G; a)$ is a either a bachelor square or a monogamous square.

Question

For which θ is $Ext_{\theta}(G; a)$ a bachelor square; for which a monogamous square?

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0	 $\theta(\delta)$	 δ		 a
÷	÷	÷	÷	
$\theta^{-1}(\delta)$		 а		 $\delta + \theta^{-1}(\delta)$
÷	÷		÷	
δ	 а			
÷	÷		÷	
	÷		а	 0
:	÷		÷	
a	$\delta + \theta(\delta)$		0	δ

Any transversal through the red *a* must pass through

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0	 $\theta(\delta)$	 δ		 a
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$ heta^{-1}(\delta)$		 а		 $\delta + \theta^{-1}(\delta)$
÷	÷		÷	
δ	 а			
÷	÷		÷	
	÷		а	 0
÷	:		÷	
\ a	$\delta + \theta(\delta)$		0	δ

Any transversal through the red *a* must pass through the red $\delta + \theta^{-1}(\delta)$.

(0	 $\theta(\delta)$	 δ		 а
	:	:	÷	÷	
	$\theta^{-1}(\delta)$		 а		 $\delta + \theta^{-1}(\delta)$
	÷	:		÷	
	δ	 а			
	÷	:		÷	
		:		а	 0
	:	:		:	
	a	$\frac{1}{\delta + \theta(\delta)}$		0	$\left \frac{\delta}{\delta} \right $

Any transversal through the red *a* must pass through the red $\delta + \theta^{-1}(\delta)$. Any transversal through the blue *a* must pass through

(0	 $\theta(\delta)$	 δ		 a
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	$\theta^{-1}(\delta)$		 а		 $\delta + heta^{-1}(\delta)$
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	δ	 а			
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		÷		а	 0
	:	:		:	
•	а	$\delta + \theta(\delta)$			δ

Any transversal through the red *a* must pass through the red $\delta + \theta^{-1}(\delta)$. Any transversal through the blue *a* must pass through the blue $\delta + \theta(\delta)$.

(0	 $\theta(\delta)$	 δ		 a
	÷	:	÷	÷	
	$\theta^{-1}(\delta)$		 а		 $\delta + heta^{-1}(\delta)$
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		:		а	 0
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-	а	$\delta + \theta(\delta)$		0	δ

Any transversal through the red *a* must pass through the red $\delta + \theta^{-1}(\delta)$. Any transversal through the blue *a* must pass through the blue $\delta + \theta(\delta)$. Any transversal through the green δ must pass through

(0	 $\theta(\delta)$	 δ		 a
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θ	$^{-1}(\delta)$		 а		 $\delta + \theta^{-1}(\delta)$
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		:		а	 0
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(-	a	$\delta + \theta(\delta)$		0	$\left \delta \right $

Any transversal through the red *a* must pass through the red $\delta + \theta^{-1}(\delta)$. Any transversal through the blue *a* must pass through the blue $\delta + \theta(\delta)$. Any transversal through the green δ must pass through the green *a*.

$\left(\begin{array}{c} 0 \end{array} \right)$		$\theta(g_s)$		<i>g</i> j		a
:		÷		:		÷
gs		а		:		÷
:				÷		÷
gi						$g_i + \theta(g_i)$
÷				:		÷
a			•••	$g_j + heta^{-1}(g_j)$	•••	$\left[\frac{\delta}{\delta} \right]$

If the three red entries are on the same transversal, then

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$\left(\begin{array}{c} 0 \end{array} \right)$		$\theta(g_s)$	 gj	 a
1		÷	:	÷
gs		а	:	÷
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gi				 $g_i + \theta(g_i)$
1 :			÷	÷
a			 $g_j + heta^{-1}(g_j)$	 δ

If the three red entries are on the same transversal, then

$$\theta(g_i) + \theta^{-1}(g_j) = g_s + \theta(g_s) + \delta.$$

(0	 $\theta(g_s)$	 <i>B</i> j	 a
	÷	÷	:	÷
	g _s	 а	:	÷
	÷		÷	÷
	gi	 		 $g_i + \theta(g_i)$
	÷		÷	÷
ĺ	а	 	 $g_j + heta^{-1}(g_j)$	 δ

If the three red entries are on the same transversal, then

$$\theta(g_i) + \theta^{-1}(g_j) = g_s + \theta(g_s) + \delta.$$

$$heta(g_i), heta^{-1}(g_j), g_s + heta(g_s) + \delta \in \{g_0, \dots, g_{m-1}\} \setminus \{0, \delta\}.$$

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An example: $Ext(\mathbb{Z}_4; a)$

1	0	1	2	3	a
	1	2	а	0	3
	2	3	0	а	1
	3	а	1	2	0
(а	0	3	1	2

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An example: $Ext(\mathbb{Z}_4; a)$

$$\theta(g_i), \theta^{-1}(g_j), g_s + \theta(g_s) + \delta \in \{1, 3\}.$$

Anthony B. Evans (Wright State University)

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An example: $Ext(\mathbb{Z}_4; a)$

$$\theta(g_i), \theta^{-1}(g_j), g_s + \theta(g_s) + \delta \in \{1, 3\}.$$

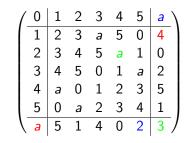
Hence

$$\theta(g_i) + \theta^{-1}(g_j) \neq g_s + \theta(g_s) + \delta.$$

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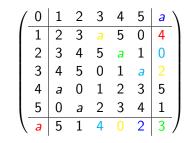
An example: $Ext(\mathbb{Z}_6; a)$



If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

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An example: $Ext(\mathbb{Z}_6; a)$

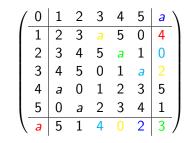


If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

Further, the cyan cells must be on the same transversal, and the yellow cells must be on the same transversal.

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An example: $Ext(\mathbb{Z}_6; a)$



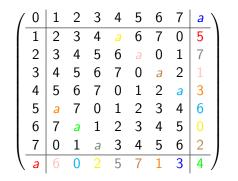
If this square has an orthogonal mate, then the red cells must be on the same transversal, the blue cells must be on the same transversal, and the green cells must be on the same transversal.

Further, the cyan cells must be on the same transversal, and the yellow cells must be on the same transversal.

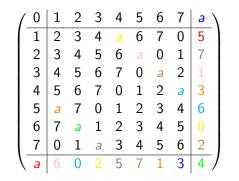
We cannot add more transversals: this is a bachelor square.

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An example: $Ext(\mathbb{Z}_8; a)$



If this square has an orthogonal mate, then cells of the same color must be on the same transversal. An example: $Ext(\mathbb{Z}_8; a)$



If this square has an orthogonal mate, then cells of the same color must be on the same transversal.

Have not determined yet if these partial transversals all complete to transversals.

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A generalization.

$$Ext(G; a_1, \dots, a_n) = \begin{pmatrix} g_0 & g_1 & \dots & g_{m-1} & a_1 & \dots & a_n \\ g_1 & & & & & & \\ \vdots & A & & B & & \\ g_{m-1} & & & & & \\ \hline a_1 & & & & & \\ \vdots & C & & D & & \\ a_n & & & & & & \end{pmatrix}$$

The g_ig_j th entry in A is either $g_i + g_j$ or one of a_1, \ldots, a_n . There are several choices for B, C, and D.

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