# Irreductible Representation of Jordan Superlgebras Kan(n)

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# Jordan Superalgebra

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*On an Algebraic Generalization of the Quantum Mechanical Formalism*, de P. Jordan, J. V. Neumann e E. Wigner [Ann. of Math. (2)35(1934), no. 1, 29-64].

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$$xy = (-1)^{|x||y|}yx$$

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- $A = M_{m+n}(F)$, $A_{\bar{0}} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $A_{\bar{1}} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$, and
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- If $A$ is an associative superalgebra and $* : A \to A$ is a superinvolution, $((a^*)^* = a$, $(ab)^* = (-1)^{|a||b|}b^*a^*)$, then the set of symetric elements $H(A, *)$ is a subsuperalgebra of $A^{(+)}$.

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- The 3-dimentional Kaplansky superalgebra, $K_3 = Fe + (Fx + Fy)$, with the multiplication: $e^2 = e$, $ex = \frac{1}{2}x$, $ey = \frac{1}{2}y$, $[x, y] = e$.
- The 1-parametric family of 4-dimensional superalgebras $D_t = (Fe_1 + Fe_2) + (Fx + Fy)$, with multiplication: $e_i^2 = e_i$, $e_1e_2 = 0$, $e_ix = \frac{1}{2}x$, $e_iy = \frac{1}{2}y$, $xy = e_1 + te_2$, $i = 1, 2$.

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V. Kac proved tha every simple finite dimentional Jordan superalgebra over a field algebraically closed of characterictic zero is isomorphic to one of the superalgebras above.

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If $J$ is a Jordan superalgebra and $V$ a super-space, then $V$ is a $J$-bimodule is the **split null extention** $E(J,V) = J \oplus V$ is Jordan superalgebra.

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A **dot-bracket superalgebra** $A = (A_0 + A_1, \cdot, \{,\})$ is an associative, supercommutative $F$-superalgebra $(A, \cdot)$ together with a super-skew-symmetric bilinear product $\{,\}$.

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$$\{f, \{g, h\}\} - \{\{f, g\}, h\} - (-1)^{|f||g|}\{g, \{f, h\}\} =$$
$$D(f) \cdot \{g, h\} + (-1)^{|g|(|f|+|h|)} D(g) \cdot \{h, f\} + (-1)^{|h|(|f|+|g|)} D(h) \cdot \{f, g\}$$

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# Jordan Superbracket

Let $A = A_0 + A_1$ be a dot-bracket superalgebra. We call $\{,\}$ a **Jordan superbracket** if:

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# Grassmann Superalgebra $G_n$

Let $G_n$ be the Grassman superalgebra with odd generators $e_1, e_2, \ldots, e_n$, with $e_i e_j + e_j e_i = 0$ and $e_i^2 = 0$.

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# Notation

$Kan(n)$ is generated as vector space by:

$$e_{i_1}e_{i_2}\cdots e_{i_k} \text{ and } \overline{e_{i_1}e_{i_2}\cdots e_{i_k}},$$

without forgetting 1 and $\overline{1}$

$$e_I := e_{i_1}e_{i_2}\cdots e_{i_k} \text{ if } I = \{i_1, i_2, \ldots, i_k\} \subseteq I_n = \{1, 2, \ldots, n\},$$

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If $I = \{i_1, \dots i_k\}$ and $J = \{j_1, \dots j_s\}$:

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where $I' = \{i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_k\}$ and $J' = \{j_1, \dots, j_{q-1}, j_{q+1}, \dots, j_s\}$.

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# Commutators

$$[R_x, R_y]_s = R_x R_y - (-1)^{|x||y|} R_y R_x.$$

**Lemma**

*Given $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_s\}$ index sets contained in $I_n = \{1, \ldots, n\}$, then*

1. $[R_{e_I}, R_{e_J}]_s = 0$, *for all* $I$ *and* $J$.
2. $[R_{e_I}, R_{\overline{e_J}}]_s = 0$, *if* $|J \cap I| \geq 2$.
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# Operators

## Lemma

*For* $Kan(n) = Kan(n)_0 + Kan(n)_1$ *and* $F$ *such that* $CarF \neq 2$*:*

1. *If* $a \in Kan(n)_1$, $a = e_I$ *or* $\overline{e_I}$, $a \neq \overline{1}$, *then:*
$$R_a^2 = 0.$$
2. *If* $a \in Kan(n)_0$, $a = e_I$ *or* $\overline{e_I}$, $a \neq 1, \overline{e_i}$, *then:*
$$R_a^3 = 0.$$
3. *If* $V$ *is irreducible and* $F$ *is algebricaly closed then:*
$$R_{\overline{1}}^2 = \alpha, \text{ for some } \alpha \in F.$$
4. $R_{\overline{e_i}}^3 = R_{\overline{e_i}}$, *for all* $i \in \{1, \ldots, n\}$.

# Special Element in $V$

**Lemma**

*If $V$ is an unital Jordan bimodulo over $Kan(n)$, then there exists $0 \neq v \in V$ such that*

$$ve_I = v\overline{e_I} = 0,$$

*for all $\phi \neq I \subseteq I_n = \{1, \ldots, n\}$.*

For example, $n = 2$

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If $I = \{i_1, \ldots, i_k\} \subseteq I_n = \{1, \ldots, n\}$ and $w \in V$:

$$w(I) := w\overline{1}\overline{e_{i_1}}\overline{1}\cdots\overline{1}\overline{e_{i_k}} := (\cdots(((w\overline{1})\overline{e_{i_1}})\overline{1})\cdots\overline{1})\overline{e_{i_k}},$$

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# Theorem: Multiplication on $V$ over $Kan(n)$

If $V$ is an unital irreducible Jordan bimodulo over the superalgebra $Kan(n)$, then $V$ is generated as vector space by the elements

$$v(I) \text{ e } \overline{v(I)}, \text{ where } I \subseteq I_n = \{1, \ldots, n\},$$

and the multiplication of $kan(n)$ over $V$ is given by:

$$v(I) \odot e_J = \begin{cases} v(I \diagdown J) & \text{if} \quad s_2 = 0 \\ 0 & \quad \text{otherwise} \end{cases}$$

$$v(I) \odot \overline{e_J} = \begin{cases} \overline{v(I \diagdown J)} & \text{if} \quad s_2 = 0 \\ 0 & \quad \text{otherwise} \end{cases}$$

$$\overline{v(I)} \odot e_J = \begin{cases} (-1)^s \overline{v(I \diagdown J)} & \text{if} \quad s_2 = 0 \\ 0 & \quad \text{otherwise} \end{cases}$$

$$\overline{v(I)} \odot \overline{e_J} = \begin{cases} (-1)^{s_1} \overline{v(I \diagdown J_1)}\,\overline{e_{J_2}} & \text{if} \quad s_2 = 1 \\ -(-1)^s \alpha(s-1) v(I \diagdown J) & \text{if} \quad s_2 = 0 \\ 0 & \quad \text{otherwise} \end{cases}$$

where $\alpha = R_{\bar{1}}^2$, $J = J_1 \cup J_2 \subseteq I_n$ com $J_1 \subseteq I$, $J_2 \cap I = \phi$, $s_p = |J_p|$ for $p = 1, 2$ and $s = s_1 + s_2 = |J|$.

# Example $n = 2$

| $1$ | $e_1$ | $e_2$ | $e_1e_2$ | $\overline{1}$ | $\overline{e_1}$ | $\overline{e_2}$ | $\overline{e_1e_2}$ |
|---|---|---|---|---|---|---|---|
| $v\overline{1}\overline{e_1}\overline{1}\overline{e_2}$ | $-v\overline{1}\overline{e_2}$ | $v\overline{1}\overline{e_1}$ | $-v$ | $v\overline{1}\overline{e_1}\overline{1}\overline{e_2}\overline{1}$ | $-v\overline{1}\overline{e_2}\overline{1}$ | $v\overline{1}\overline{e_1}\overline{1}$ | $-v\overline{1}$ |
| $v\overline{1}\overline{e_2}$ | $0$ | $v$ | $0$ | $v\overline{1}\overline{e_2}\overline{1}$ | $0$ | $v\overline{1}$ | $0$ |
| $v\overline{1}\overline{e_1}$ | $v$ | $0$ | $0$ | $v\overline{1}\overline{e_1}\overline{1}$ | $v\overline{1}$ | $0$ | $0$ |
| $v$ | $0$ | $0$ | $0$ | $v\overline{1}$ | $0$ | $0$ | $0$ |
| $v\overline{1}\overline{e_1}\overline{1}\overline{e_2}\overline{1}$ | $v\overline{1}\overline{e_2}\overline{1}$ | $-v\overline{1}\overline{e_1}\overline{1}$ | $-v\overline{1}$ | $\alpha v\overline{1}\overline{e_1}\overline{1}\overline{e_2}$ | $0$ | $0$ | $\alpha v$ |
| $v\overline{1}\overline{e_2}\overline{1}$ | $0$ | $-v\overline{1}$ | $0$ | $-\alpha v\overline{1}\overline{e_2}$ | $-v\overline{1}\overline{e_1}\overline{1}\overline{e_2}$ | $0$ | $v\overline{1}\overline{e_1}$ |
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| $v\overline{1}$ | $0$ | $0$ | $0$ | $\alpha v$ | $v\overline{1}\overline{e_1}$ | $v\overline{1}\overline{e_2}$ | $0$ |

where $va_1a_2\dots a_p := (\dots((va_1)a_2)\dots)a_p$ and $\alpha = R_{\overline{1}}^2$.

If $\alpha = 0$ then we have the regular bimodule.

# Example $n = 2$

| $1$ | $e_1$ | $e_2$ | $e_1e_2$ | $\bar{1}$ | $\overline{e_1}$ | $\overline{e_2}$ | $\overline{e_1e_2}$ |
|---|---|---|---|---|---|---|---|
| $v\bar{1}\overline{e_1}\bar{1}\overline{e_2}$ | $-v\bar{1}\overline{e_2}$ | $v\bar{1}\overline{e_1}$ | $-v$ | $v\bar{1}\overline{e_1}\bar{1}\overline{e_2}\bar{1}$ | $-v\bar{1}\overline{e_2}\bar{1}$ | $v\bar{1}\overline{e_1}\bar{1}$ | $-v\bar{1}$ |
| $v\bar{1}\overline{e_2}$ | $0$ | $v$ | $0$ | $v\bar{1}\overline{e_2}\bar{1}$ | $0$ | $v\bar{1}$ | $0$ |
| $v\bar{1}\overline{e_1}$ | $v$ | $0$ | $0$ | $v\bar{1}\overline{e_1}\bar{1}$ | $v\bar{1}$ | $0$ | $0$ |
| $v$ | $0$ | $0$ | $0$ | $v\bar{1}$ | $0$ | $0$ | $0$ |
| $v\bar{1}\overline{e_1}\bar{1}\overline{e_2}\bar{1}$ | $v\bar{1}\overline{e_2}\bar{1}$ | $-v\bar{1}\overline{e_1}\bar{1}$ | $-v\bar{1}$ | $\alpha v\bar{1}\overline{e_1}\bar{1}\overline{e_2}$ | $0$ | $0$ | $\alpha v$ |
| $v\bar{1}\overline{e_2}\bar{1}$ | $0$ | $-v\bar{1}$ | $0$ | $-\alpha v\bar{1}\overline{e_2}$ | $-v\bar{1}\overline{e_1}\bar{1}\overline{e_2}$ | $0$ | $v\bar{1}\overline{e_1}$ |
| $v\bar{1}\overline{e_1}\bar{1}$ | $-v\bar{1}$ | $0$ | $0$ | $\alpha v\bar{1}\overline{e_1}$ | $0$ | $v\bar{1}\overline{e_1}\bar{1}\overline{e_2}$ | $-v\bar{1}\overline{e_2}$ |
| $v\bar{1}$ | $0$ | $0$ | $0$ | $\alpha v$ | $v\bar{1}\overline{e_1}$ | $v\bar{1}\overline{e_2}$ | $0$ |

where $va_1a_2 \dots a_p := (\dots((va_1)a_2)\dots)a_p$ and $\alpha = R_1^2$.

If $\alpha = 0$ then we have the regular bimodule.

# Thanks

THANKS!