Non Moufang Loop Satisfying Moufang's Theorem

Maria de Lourdes Merlini Giuliani
work in progress with Giliard dos Anjos

Federal University of ABC

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MOTIVATION

Moufang's Theorem says that if 3 elements associate in some order, then they associate in any order.

Moufang loops satisfies Moufang's Theorem.

Is there other category that satisfy Moufang's Theorem?

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THE QUESTION

Is every variety of loops that satisfies Moufang's Theorem contained in the variety of Moufang loops?

FUNDAMENTALS

A *quasigroup* is a pair $(Q, .)$ such that if in the equation $a.b = c$ any two of the elements are known, then the third is uniquely determined.

A *loop* is a quasigroup with identity element, say 1.

A *Moufang loop* is a loop satisfying the identity $(xy)(zx) = (x(yz))x$

Moufang Theorem: If $(a, b, c) = 1$ for some $a, b, c \in Q$, where Q is a loop, then a, b, c generate a subgroup of Q.

A loop Q is a *Steiner loop* if Q has *Inverse Property*, $(x^{-1}(xy) = y)$ and of exponent 2.

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From now on I will say that a loop L satisfies the property $\mathcal{P}$ if:
L is not Moufang loop, but satisfies Moufang's theorem

Steiner loops are not Moufang loops.

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An example of a Steiner loop with property $\mathcal{P}$

1	2	3	4	5	6	7	8	9	10
2	1	4	3	8	10	9	5	7	6
3	4	1	2	10	9	8	7	6	5
4	3	2	1	9	8	10	6	5	7
5	8	10	9	1	7	6	2	4	3
6	10	9	8	7	1	5	4	3	2
7	9	8	10	6	5	1	3	2	4
8	5	7	6	2	4	3	1	10	9
9	7	6	5	4	3	2	10	1	8
10	6	5	7	3	2	4	9	8	1

It has this property:

For any x, y, z, such that $x \neq y$; $y \neq z$; $z \neq x$,
if $x.yz = xy.z$ then $z = xy$
Then $< x, y, z > = < x, y >$

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7	9	8	10	6	5	1	3	2	4
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Second example: A Steiner loop of order 16 ($(16,80)$ in GAP)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	9	5	10	6	8	13	15	11	16	12	14
4	3	2	1	11	8	12	6	16	14	5	7	15	10	13	9
5	6	7	11	1	2	3	13	14	15	4	16	8	9	10	12
6	5	9	8	2	1	15	4	3	11	10	14	16	12	7	13
7	8	5	12	3	15	1	2	13	16	14	4	9	11	6	10
8	7	10	6	13	4	2	1	12	3	16	9	5	15	14	11
9	10	6	16	14	3	13	12	1	2	15	8	7	5	11	4
10	9	8	14	15	11	16	3	2	1	6	13	12	4	5	7
11	12	13	5	4	10	14	16	15	6	1	2	3	7	9	8
12	11	15	7	16	14	4	9	8	13	2	1	10	6	3	5
13	14	11	15	8	16	9	5	7	12	3	10	1	2	4	6
14	13	16	10	9	12	11	15	5	4	7	6	2	1	8	3
15	16	12	13	10	7	6	14	11	5	9	3	4	8	1	2
16	15	14	9	12	13	10	11	4	7	8	5	6	3	2	1

In Van Lint & Wilson (2001) we find a construction of Steiner Triple System:

Let $n = 2t + 1$ and define $Q := \mathbb{Z}_n \times \mathbb{Z}_3$. All triples

$\{(x, i), (y, i), (\frac{x+y}{2}, i + 1)\}$, where $x \neq y$ in $\mathbb{Z}_n$, provides STS.

Let $L = Q \cup \{e\}$, then $(L, *)$ is a Steiner loop, $|L| = 3n + 1$, with the rules:

$(x, i) * (y, i) = (\frac{x+y}{2}, i + 1)$

$(x, i) * (y, i + 1) = (2y - x, i)$

$(x, i) * (y, i - 1) = (2x - y, i - 1)$

$(x, i) * (x, i) = e$

for any $x \neq y \in \mathbb{Z}_n$.

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We verified that if L is a Steiner loop constructed as above with $|L| = k$, for

$$k \in \{28, 34, 40, 46, 52, 58, 79, 76, 82, 88, 94, 100, 112, 118, 124, 130, 136, 142, 154\}$$

then L has property $\mathcal{P}$.

For $k = 22, 64, 106, 148,$ such L doesnt have property $\mathcal{P}$

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Theorem 1

Let L be a Steiner loop constructed as above.
If 7 is an invertible element in $\mathbb{Z}_n$, then L has the property $\mathcal{P}$.

Note: If 7 is not invertible in $\mathbb{Z}_n$, then there exists $a \in \mathbb{Z}_n$
$7a = 0 \pmod{n}$.

The associator $((0,1),(0,0),(a,0)) = 1$

whereas $((0,1),(a,0),(0,0)) \neq 1$.

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Theorem 2

Let L be a Steiner loop satisfying $\mathcal{P}$ and let M be a Moufang loop. Then $L \times M$ satisfies $\mathcal{P}$.

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VARIETY $\mathcal{V}$

Define $\mathcal{V}$ as the variety of loops satisfying $\mathcal{P}$.

$\mathcal{V}$= { Moufang loop $\times$ Steiner loops satisfying $\mathcal{P}$ }

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THANK YOU!