Simple Right Conjugacy Closed Loops

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Mile High Conference
11 August 2013

Definition

For a loop Q, we define:

left and right translations of a by x	$aL_x = xa \qquad aR_x = ax$
right section of Q	$R_Q = \{R_x \mid x \in Q\}$
right multiplication group of Q	$\mathrm{Mlt}_\rho(Q) = \langle R_Q \rangle$
multiplication group of Q	$\mathrm{Mlt}(Q) = \langle L_x, R_x \mid \forall x \in Q \rangle$
inner mapping group of Q	$\mathrm{Inn}(Q) = \{\theta \in \mathrm{Mlt}(Q) \mid 1\theta = 1\}$

Definition

A subset S of a group G is *closed under conjugation* if $x^{-1}yx \in S$ for all $x, y \in S$.

Defintion

A loop Q is a *right conjugacy closed loop* (or RCC loop) if R_Q is closed under conjugation.

Note: $R_x^{-1}R_yR_x \in R_Q$ for all $x, y \in Q$.

Proposition

For a loop Q, the following are equivalent:

(1) Q is an RCC loop,

(2) The following holds for all $x, y, z \in Q$:

$$R_x^{-1} R_y R_x = R_{x \backslash yx}. \tag{RCC$_1$}$$

(3) The following holds for all $x, y, z \in Q$:

$$(xy)z = (xz) \cdot z \backslash (yz). \tag{RCC$_2$}$$

Definition

For a loop Q, a subset S of Q is a subloop if $(S, \cdot, \backslash, /)$ is a loop. A subloop N of a loop Q is a *normal subloop*, $N \trianglelefteq Q$, if it is invariant under $\mathrm{Inn}(Q)$.

Definitions

the left nucleus of Q,	$N_\lambda(Q) = \{a \in Q \mid a \cdot xy = ax \cdot y \ \forall x, y \in Q \}$,
the middle nucleus of Q,	$N_\mu(Q) = \{a \in Q \mid x \cdot ay = xa \cdot y \ \forall x, y \in Q \}$,
the right nucleus of Q,	$N_\rho(Q) = \{a \in Q \mid x \cdot ya = xy \cdot a \ \forall x, y \in Q \}$,
the nucleus of Q,	$N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$,
the commutant of Q,	$C(Q) = \{a \in Q \mid xa = ax \ \forall x \in Q\}$,
the center of Q,	$Z(Q) = N(Q) \cap C(Q)$.

Proposition

Let Q be a RCC loop. Then

(i) $N_\mu(Q) = N_\rho(Q) \trianglelefteq Q$ and

(ii) $C(Q) \leq N_\lambda(Q)$.

Note:

Let Q be a RCC-loop with $N \trianglelefteq Q$ and consider $R_N = \{R_x \mid x \in N\}$. Fix $x \in N$ and then $\forall y \in Q$, $R_y R_x R_y^{-1} = R_{(yx/y)} \in R_N$ since $yx/y \in N$. Hence, normal subloops of Q correspond to unions of conjugacy classes in R_Q.

Notation

Let $\mathbb{F}_q$ be the finite field of order where $q = p^n$ for a prime p and some $n > 0$. For a matrix M, let

$Det(M)$ denote the *determinant of the matrix* M,

$Tr(M)$ denote the *trace of the matrix* M and

$Char(M)$ denote the *characteristic polynomial of the matrix* M.

All matrices will be of size 2×2 (*i.e.* $M \in GL(2, q)$), hence

$$Char(M) = x^2 - Tr(M)x + Det(M) \in \mathbb{F}_q[x].$$

Setup

First, let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_q[x]$. For each $b \in \mathbb{F}_q$, define

$$M_{(0,b)} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

and for $a \neq 0$,

$$M_{(a,b)} = \begin{pmatrix} r - b & \frac{f(b)}{-a} \\ a & b \end{pmatrix}.$$

Lemma

Let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_q[x]$. The conjugacy class of all matrices in $GL(2, q)$ with characteristic polynomial $f(x)$ is precisely the set $\{M_{(a,b)} \mid a, b \in \mathbb{F}_q\}$ for $a \neq 0$.

Theorem (MG)

Let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_p[x]$. Let $Q = \mathbb{F}_q^2 \backslash \{[0, 0]\}$, written as a set of row vectors. Define a binary operation $\circ_f$ on Q by

$$[a, b] \circ_f [c, d] = [a, b] M_{(c,d)}.$$

Then $(Q, \circ_f)$ is a loop.

Note

In $(Q, \circ_f)$, we have

(i) $[a, b] \circ_f [c, d] = [a(r - d) + bc, \frac{-af(d)}{c} + bd] \qquad c \neq 0,$

(ii) $[a, b] \circ_f [c, d] = [ad, bd] \qquad c = 0,$

Notation

In $(Q, \circ_f)$,

(i) $[x, y]$ denotes an element in Q,

(ii) $R_{[x,y]}$ denotes the right translation by $[x, y]$,

(iii) $M_{(x,y)}$ denotes the matrix associated with the right translation by $[x, y]$.

Lemma (MG)

In $(Q, \circ_f)$

(i) for $a \neq 0$, $R_{[a,b]}^{-1} = M_{(a,b)}^{-1} = \begin{pmatrix} r-b & \frac{f(b)}{-a} \\ a & b \end{pmatrix}^{-1} = \frac{1}{s}\begin{pmatrix} b & f(b)/a \\ -a & r-b \end{pmatrix} = \frac{1}{s} M_{[-a,r-b]}$,

(ii) $R_{[0,b]}^{-1} = \frac{1}{b}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

(iii) $R_{[a,b],[c,d]} = M_{(a,b)} M_{(c,d)} M_{[a,b]\circ_f[c,d]}^{-1} =$
$\begin{pmatrix} s & \frac{-(a^2 sf(d) - abcds - abcd + abcr + acdr - acr^2 + acrs + c^2 f(b))}{(ac(bc-ad+ar))} \\ 0 & 1 \end{pmatrix}$,

(iv) $R_{[a,b],[0,d]} = M_{(a,b)} M_{(0,d)} M_{[a,b]\circ_f[0,d]}^{-1} = \begin{pmatrix} d^2 & \frac{(d-1)(b-r+bd)}{a} \\ 0 & 1 \end{pmatrix}$,

(v) $R_{[0,b],[c,d]} = M_{(0,b)} M_{(c,d)} M_{[0,b]\circ_f[c,d]}^{-1} = \begin{pmatrix} b^2 & \frac{(b-1)(d-r+bd)}{c} \\ 0 & 1 \end{pmatrix}$ and

(vi) $R_{[0,b],[0,d]} = M_{(0,b)} M_{(0,d)} M_{[0,b]\circ_f[0,d]}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition

Let Q be a loop. Then $a \in C(Q) \cap N_\lambda(Q) \Leftrightarrow R_a \in Z(Mlt_\rho(Q))$.

Lemma

$C(Q, \circ_f) = \{[0, b] \mid \forall b \in \mathbb{F}_q \; b \neq 0\}$. That is, the only elements of $C(Q, \circ_f)$ are in the set $\{R_{[a,b]} \mid [a, b] \in C(Q, \circ_f)\}$.

Theorem (MG)

$(Q, \circ_f)$ is an RCC loop.

Lemma (MG)

Let $q \neq 3$. Then $C(Q, \circ_f) = N_\lambda(Q, \circ_f)$. If $q = 3$ and $r \neq 0$, then $C(Q, \circ_f) = N_\lambda(Q, \circ_f)$.

Note

As noted before, normal subloops of Q correspond to unions of conjugacy classes of matrices in GL(2,q) which are contained in $R_{(Q,\circ_f)}$. $R_{(Q,\circ_f)}$ itself is the union of conjugacy classes, namely, $\{M_{(a,b)} | a, b \in Q, a, b \neq 0\}$, which has size $q^2 - q$, and the $q - 1$ one-element conjugacy classes in the center of $GL(2, q)$. Since the order of a normal subloop of Q must divide $|Q| = q^2 - 1$.

Lemma (MG)

The only non-trivial normal subgroups of $(Q, \circ_f)$ are $C(Q, \circ_f)$ and $\{[0, 1], [0, -1]\}$.

Theorem (MG)

Let $f(x) = x^2 - rx + s$ be irreducible.

(i) If $r \neq 0$, then $(Q, \circ_f)$ is simple.

(ii) If $r = 0$, then $Z(Q, \circ_f) = \{[0, \pm 1]\}$ and $(Q, \circ_f)/Z(Q, \circ_f)$ is simple.

Elements

Let $q = 3$ and hence the elements of $(Q, \circ_f)$ are

$$\{[0,1],[0,2],[1,0],[1,1],[1,2],[2,0],[2,1],[2,2]\}.$$

Conjugacy Class

Let $f(x) = x^2 + 2x + 2$ and note that $f(x)$ is irreducible in $\mathbb{F}_3$.

$$\left\{\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\1 & 1\end{pmatrix},\begin{pmatrix}2 & 2\\1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix},\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix},\begin{pmatrix}2 & 1\\2 & 2\end{pmatrix}\right\}.$$

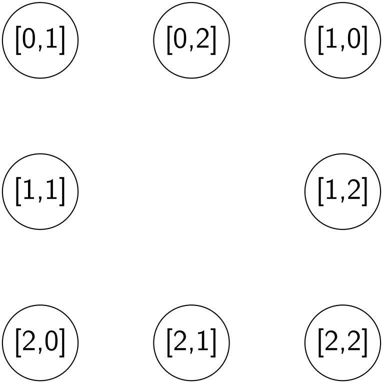
Full Set of Matrices

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix},\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix},\begin{pmatrix}2 & 2\\ 1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix},\begin{pmatrix}0 & 2\\ 2 & 1\end{pmatrix},\begin{pmatrix}2 & 1\\ 2 & 2\end{pmatrix}\right\},$$

Note

$$M_{(0,1)} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, M_{(0,2)} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} M_{(1,0)} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \dots$$

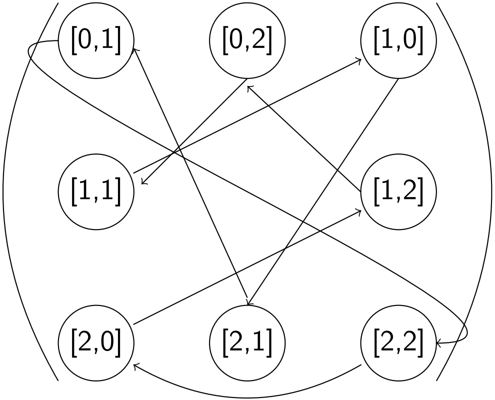
Visualizing the construction



Visualizing the construction

$$\begin{pmatrix} [0,1] & [0,2] & [1,0] \\ [1,1] & & [1,2] \\ [2,0] & [2,1] & [2,2] \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

Visualizing the construction



$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

Right Section

$$R_{(Q,\circ_f)} = \{(), (1,2)(3,6)(4,8)(5,7), (1,3,4,7,2,6,8,5), (1,4,5,6,2,8,7,3),$$
$$(1,5,3,8,2,7,6,4), (1,6,7,4,2,3,5,8), (1,7,8,3,2,5,4,6), (1,8,6,5,2,4,3,7)\}.$$

Loop $(Q, \circ_f)$

$\circ_f$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	6	8	7	3	5	4
3	3	6	4	1	8	5	2	7
4	4	8	7	5	1	2	6	3
5	5	7	1	6	3	8	4	2
6	6	3	8	2	4	7	1	5
7	7	5	2	3	6	4	8	1
8	8	4	5	7	2	1	3	6

Table: Multiplication Table for $(Q, \circ_f)$

Simple RCC Loops

q	Order	Number of primitive polynomials	Number of non-isomorphic, nonassociative RCC Loops	Number of Simple RCC loops	Exhaustive
3	8	3	3	2	✓
5	12	2	2	2	✓
4	15	6	3	3	✓
5,7	24	10,3	13	11	
9	40	2	2	2	
7	48	21	21	18	
11	60	5	5	5	
8	63	28	10	10	
9	80	36	18	16	
13	84	6	6	6	
11	120	55	55	50	
13	168	78	78	72	
16	255	120	30	30	

Conjecture 1

For $(Q, \circ_f)$

$$\mathrm{Inn}_\rho(Q, \circ_f) = \{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x = a^2 s^m \quad a, y \in \mathbb{F}_q \quad m \in \mathbb{Z}\}.$$

Conjecture 2

Let p be a prime number.

(i) If $q = p$, then the number of nonisomorphic RCC loops constructed from $GL(2, q)$ is $\frac{q^2-q}{2}$.

(ii) If $q = p^2$, then the number of nonisomorphic RCC loops constructed from $GL(2, q)$ is $\frac{q^2-q}{4}$.