# Finite commutative automorphic loops

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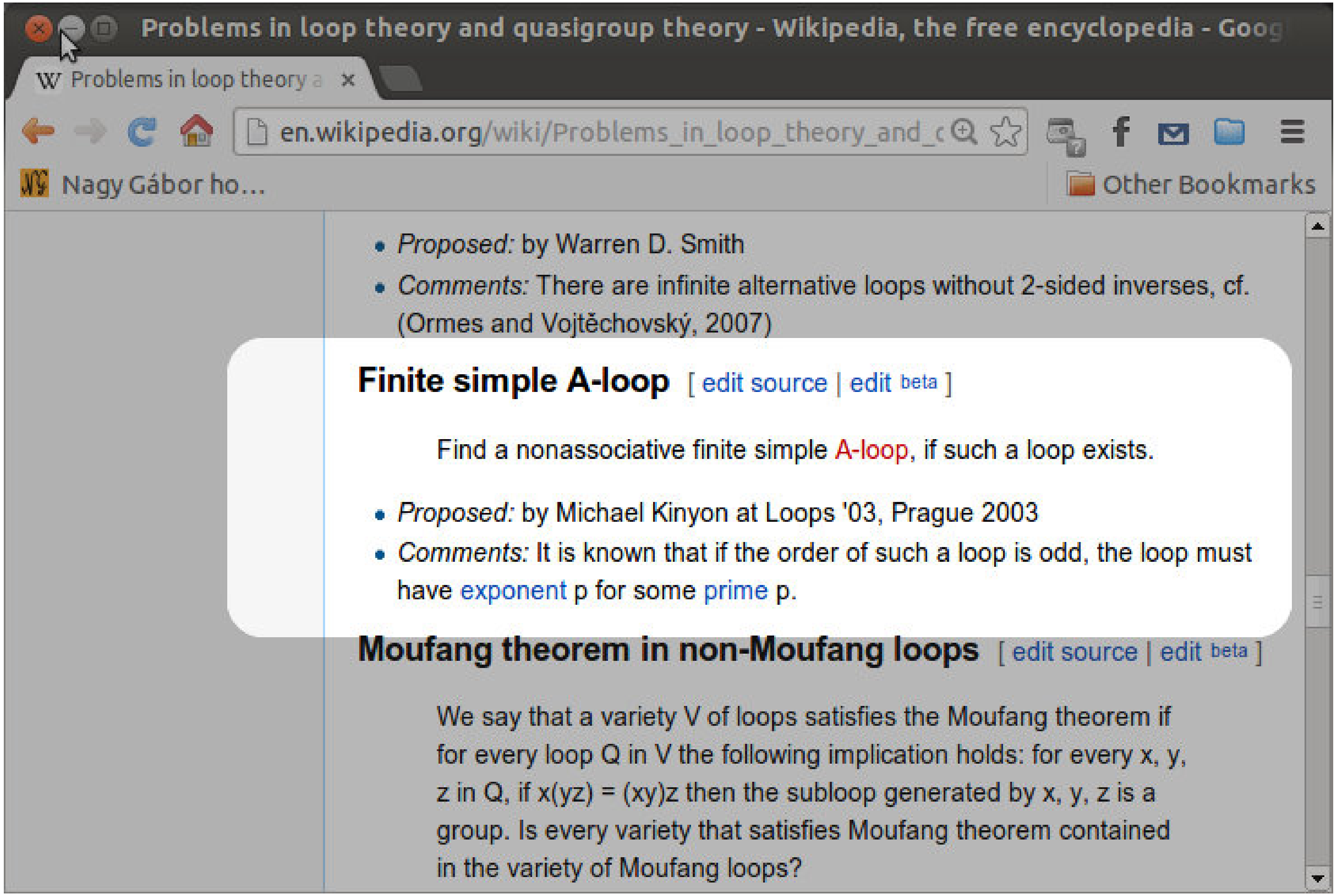
# Overview

1. Automorphic loops
2. Finite simple commutative A-loops
3. Centerless finite commutative A-loops of exponent 2

# Basic concepts

- Left and right multiplication maps: $L_x, R_x$
- Multiplication group, inner mapping group: $\mathrm{Mlt}(Q), \mathrm{Inn}(Q)$
- Normal subloop, center, factor loop
- Solvable loop, simple loop
- Automorphic loop (A-loop): $\mathrm{Inn}(Q) \leq \mathrm{Aut}(Q)$
- Power-associativity, $p$-loop
- Anti-automorphic inverse property: $(xy)^{-1} = y^{-1}x^{-1}$

# Problem setting

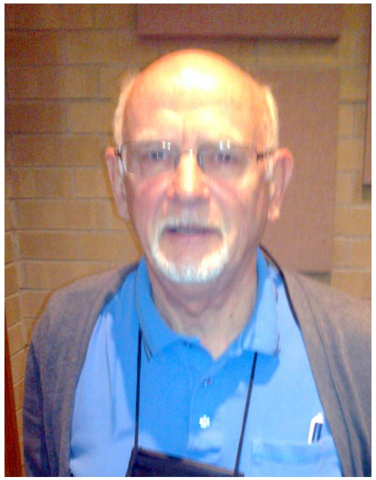


# Previous work done by

Přemysl

Michael

Petr



Ken

J. D.

Piroska

# Previous results

**Theorem (Albert 1943)**

$Q$ is simple if and only if $\mathrm{Mlt}(Q)$ acts primitively on $Q$.

**Proposition**

Let $Q$ be a finite, commutative automorphic loop.

1. $Q \cong O(Q) \times E(Q)$ where $|O(Q)|$ is odd and $E(Q)$ is a 2-loop.
2. If $|Q|$ is odd, then $Q$ is solvable.
3. If $Q$ is simple, then $Q$ has exponent 2.

**Proposition**

Every A-loop has the anti-automorphic inverse property: $JL_xJ = R_x^{-1}$.

# Main result

**Main theorem (Grishkov, Kinyon, N 2012)**

1. Every finite, commutative automorphic loop is solvable.
2. Every automorphic 2-loop is solvable.

Main strategies:

- Heavy equational computations, by hand and Prover9.
- Deep results from the theory of finite groups.
- Deep results from the theory of Lie rings.

**Proposition**

Every finite simple, automorphic 2-loop is commutative of exponent 2.

# Commutative A-loops of exponent 2 and Lie rings

- Let $(Q, \circ)$ be a finite simple commutative A-loop of exponent 2.
- Using deep group theory, we show that $\mathrm{Mlt}(Q)$ is a primitive group of affine type.
- This defines an elementary abelian 2-group operation $(Q, +)$.
- Define the bracket $[x, y] = x + y + x \circ y$.
- (Wright 1969) $(Q, +, [\cdot, \cdot])$ is a simple nonassociative algebra over $\mathbb{F}_2$.

**Proposition**

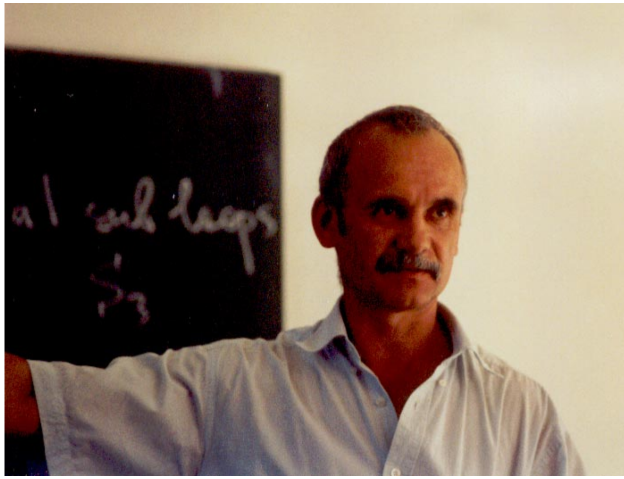
$(Q, +, [\cdot, \cdot])$ is a simple Lie algebra over $\mathbb{F}_2$ satisfying

$$\mathrm{ad}(u)\,\mathrm{ad}([u, w]) = 0$$

for all $u, w \in Q$.

# The "crust of a thin sandwich"

Efim Zel'manov



Grishkov at LOOPS'03

**Theorem (Zel'manov,Kostrikin 1990)**

Let $\mathfrak{g}$ be a Lie ring generated by a finite collection of elements $a$ satisfying $\mathrm{ad}(a)^2 = 0$ and $\mathrm{ad}(a)\,\mathrm{ad}(x)\,\mathrm{ad}(a) = 0$ for all $x \in \mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent.

*Proof of the Main Theorem.* Some more juggling with equations shows that ZK applies for $(Q, +, [\cdot,\cdot])$, a contradiction. $\square$

# Automorphic $p$-loops and nilpotency

**Previous results (Jedlička, Kinyon, Vojtěchovský, Csörgő 2010/11)**

1. For $p > 2$, commutative automorphic $p$-loops are nilpotent.
2. There are nonnilpotent automorphic $p$-loops for $p > 2$.
3. There are nonnilpotent commutative automorphic 2-loops.

# The construction

**Theorem 1**

Let $n, m$ be positive integers, $\beta : \mathbb{F}_2^m \to \mathbb{F}_2^{n\times n}$ a linear map such that $y, y' \in \mathbb{F}_2^m$

1. $I + \beta(y)$ is nonsingular, and
2. $\beta(y)$, $\beta(y')$ commute.

Define the operation $\circ$ on $\mathbb{F}_2^{n+m}$ by

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + x_2 + \beta(y_1)x_2 + \beta(y_2)x_1, y_1 + y_2). \quad (1)$$

Then $(\mathbb{F}_2^{n+m}, \circ, 0)$ is a commutative $A$-loop of exponent 2.

Note that with

$$[(x_1, y_1), (x_2, y_2)] = [\beta(y_1)x_2 + \beta(y_2)x_1, 0],$$

$(\mathbb{F}_2^{n+m}, +, [\cdot,\cdot])$ is a solvable Lie algebra of characteristic 2.

# Centerless commutative A-loops of exponent 2

## Theorem 2

$(\mathbb{F}_2^{n+m}, \circ)$ has trivial center if and only if $\beta(y)$ is nonsingular for all $y \in \mathbb{F}_2^m$. In particular, in the centerless case $m < n$.

## Example

Let $m < n$, $m \mid n$ and fix $\alpha \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^m}$. Define $\beta$ by

$$\beta(y)x = \alpha xy.$$

Then the corresponding comm. $A$-loop of exp. 2 has trivial center.

Remarks.

1. Additively closed subsets of $GL(n, p) \cup \{0\}$ are interesting on their own. [$\rightsquigarrow$ partial additive spreads]
2. **PROBLEM:** $\exists$ infinite simple comm. A-loops of exp. 2 ???

# The End

**THANK YOU FOR YOUR ATTENTION!**