The Structure of (Power Associative) Conjugacy Closed Loops

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Left Nucleus: $\mathrm{N}_\lambda(Q) = \{a : a \cdot xy = ax \cdot y\}$

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Nucleus: $\mathrm{N}(Q) = \mathrm{N}_\lambda(Q) \cap \mathrm{N}_\mu(Q) \cap \mathrm{N}_\rho(Q)$

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Conjugacy Closed (CC):

$$\text{LCC} + \text{RCC}$$

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Extra: $x(y \cdot zx) = (xy \cdot z)x$

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Theorem (Kinyon, P.)

In PACC-loops, x^6 is nuclear.

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Note: $S(L) \cap C(L) = \mathrm{N}(L)$.

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Let L be a PACC-loop.
(i) If a is in $S(L)$*, then* $(a, x, y)^4 = 1$
(ii) If b is in $C(L)$*, then* $(b, x, y)^3 = 1$
(iii) $(x, y, z)^{12} = 1$

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Note: We do not know if the 4 in (i) is sharp.

Theorem (Kinyon, P.)

Sylow p-subloops exist for finite PACC-loops.