The exceptions that prove the rule

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EXCEPTION

EXCEPTION

Classification of real finite-dimensional algebras & other systems

SYSTEM	NORMAL	EXCEPTIONAL
Normed division algebra	$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$	$\mathbb{O}$
simple Alternative algebra	Associative $\mathbb{K}^{n\times n}$	$\mathbb{O}$
simple compact Lie algebra	$\mathfrak{ah}(n, \mathbb{K})$ $= \{X \in \mathbb{K}^{n\times n} : X^{\dagger} = -X\}$	G_2, F_4, E_6, E_7, E_8
simple compact classical Lie superalgebra	$\mathfrak{osp}(m,n;\mathbb{K})$ $P(n,\mathbb{K})$ $Q(n,\mathbb{K})$	$D(2,1;\alpha)$ $G(3)$, $F(4)$
simple compact Jordan algebra	"Special" $\mathbb{K}^{n+1}$ $H_n(\mathbb{K}) = \{X \in \mathbb{K}^{n\times n} : X^{\dagger} = X\}$	$H_3(\mathbb{O})$
irreducible manifold Projective space	Desarguian $P^n(\mathbb{K})$	$P^2(\mathbb{O})$

$3\infty + 5$ AGAIN

Regular polytope	Plane polygons $\{p\}$ α_n (simplex) β_n (cross) γ_n (cube)	$\{5,3\}$ (dodecahedron) $\{3,5\}$ (icosahedron) $\{5,3,3\}$ $\{3,4,3\}$ $\{3,3,5\}$

ALGEBRAS

Lie algebra

Anti-commutative, non-associative bracket

modelled on commutator (Jacobi identity)

Jordan algebra

Commutative, non-associative product

modelled on anti-commutator

e.g. hermitian matrices $H_n(\mathbb{K})$

$(n = 2 \text{ or } 3 \text{ if } \mathbb{K} = \mathbb{O})$

Composition algebra

Multiplicative quadratic form

e.g. $Q(x) = x\bar{x}$

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Composition algebra

Multiplicative quadratic form $Q(xy) = Q(x)Q(y)$

e.g. $Q(x) = x\bar{x}$

JORDAN ALGEBRAS & PROJECTIVE SPACES

Jordan algebras

$$x \cdot y = y \cdot x$$
$$x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y)$$

e.g. $x \cdot y = \frac{1}{2}(xy + yx)$ in an associative algebra A. Such a Jordan algebra (denoted A^+), or a subalgebra of such, is called special. (i.e. ordinary)

algebras 1.jpg

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Examples: 1. Inner-product vector space V

$$J_\pm(V) = \mathbb{R} \oplus V \quad \text{with} \quad x \cdot y = \pm \langle x, y \rangle$$

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2. Hermitian matrices

$$H_n(\mathbb{K}) = \{ X \in \mathbb{K}^{n \times n} : X^\dagger = X \}$$

algebras 3.jpg

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Associated with projective space $\mathbb{P}^{n-1}(\mathbb{K})$:

$\underset{\sim}{x} \in \mathbb{K}^n$		$X = \underset{\sim}{x}\,\underset{\sim}{x}^\dagger \in H_n(\mathbb{K})$
ray $\{\underset{\sim}{x}\, u : u \in \mathbb{K}\}$ in $\mathbb{P}^{n-1}(\mathbb{K})$	$\longleftrightarrow$	idempotent $(X^2 = X)$
normalised $\underset{\sim}{x}^\dagger \underset{\sim}{x} = 1$		normalised $\operatorname{tr} X = 1$

Projective transf^n's $\underset{\sim}{x} \longmapsto M\underset{\sim}{x}$ $\qquad$ $X \longmapsto M X M^\dagger$

Infinitesimally, $\delta \underset{\sim}{x} = \varepsilon A \underset{\sim}{x}$ $\qquad$ $\delta X = \varepsilon \left(A X + X A^\dagger \right)$

algebras 4.jpg

CLASSIFICATION OF ALGEBRAS 1

Associative algebras

The simple associative algebras over a field F are all matrix algebras $D^{n \times n}$ where D is an associative division algebra over F.

(Artin - Wedderburn theorem)

CLASSIFICATION OF ALGEBRAS 2

Frobenius (1878): The only associative division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$

Hurwitz (1891): The only composition algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

Bruck & Kleinfeld (1952): The only alternative division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

Bott & Kervaire, Milnor (1958): The only division algebras have dimension 1, 2, 4, 8

algebras.jpg

CLASSIFICATION OF ALGEBRAS 3

Jordan algebras (Jordan, von Neumann, Wigner)

All finite-dimensional Jordan algebras are direct sums of simple ideals, which can only be

$$J(V) = V \oplus \mathbb{R} \qquad V = \mathbb{R}^{m,n}$$

or $$H_n(\mathbb{K}) = \{ X \in \mathbb{K}^{n \times n} : \quad X^\dagger = X \}$$

where $\mathbb{K}$ is a composition algebra

& $n \leq 3$ if $\mathbb{K} = \mathbb{O}$.

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All but $H_3(\mathbb{O})$ lie in associative algebras.

Isomorphisms: $H_2(\mathbb{K}) \cong J(\mathbb{K} \oplus \mathbb{R})$

Jacobson: similar theory to Artin-Wedderburn

CLASSIFICATION OF LIE ALGEBRAS

Lie algebras over $\mathbb{C}$

The simple Lie algebras over $\mathbb{C}$ are:

A_n  $\mathfrak{sl}(n+1)$

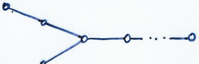
B_n  $\mathfrak{so}(2n+1)$

C_n  $\mathfrak{sp}(2n)$

D_n  $\mathfrak{so}(2n)$

G_2 

F_4 

$E_{6,7,8}$ 

CLASSIFICATION OF LIE ALGEBRAS

Simple Lie algebras over $\mathbb{C}$

(some overlap)

Cartan – Killing: 4 families a_n, b_n, c_n, d_n

5 exceptions g_2, f_4, e_6, e_7, e_8

Matrix models: 3 families $sl(n)$, $so(n)$, $sp(n)$

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Compact simple Lie algebras over $\mathbb{R}$

$su(n, \mathbb{K})$ over normed division algebra $\mathbb{K}$

4 exceptions g_2, e_6, e_7, e_8

cf simple associative algebras (Wedderburn):

matrix algebras $M_n(\mathbb{K})$ over division algebra $\mathbb{K}$

Compact simple Lie algebras over $\mathbb{R}$

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4 exceptions g_2, e_6, e_7, e_8

cf simple associative algebras (Wedderburn):

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Simple Lie algebras over $\mathbb{R}$

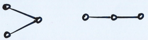
$sl(n, \mathbb{K})$, $so(n, \mathbb{K})$, $sp(n, \mathbb{K})$, $su(m, n; \mathbb{K})$

1 exception e_8

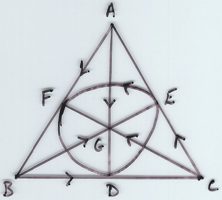
Most isomorphisms e.g. $sl(2, \mathbb{K}) = so(\dim \mathbb{K} + 1, 1)$

algebras 4.jpg

ISOMORPHISMS BETWEEN LIE ALGEBRAS

$A_1 \cong B_1 \cong C_1$ 	$su(2) \cong so(3) \cong sp(1)$	$sl(2,\mathbb{R}) \cong so(2,1)$
$B_2 \cong C_2$ 	$so(5) \cong sp(2)$	$so(3,2) \cong sp(4,\mathbb{R})$
$D_2 \cong A_1 \oplus A_1$ 	$so(4) \cong su(2) \oplus su(2)$	$so(3,1) \cong sl(2,\mathbb{C})$
$D_3 \cong A_3$ 	$so(6) \cong su(4)$	$so(4,2) \cong su(2,2)$ $so(5,1) \cong sl(2,\mathbb{H})$

OCTONIONS



Basis $1, e_A, \ldots, e_G$ with multiplication

$$e_P e_Q = e_R = -e_Q e_P$$

where PQR is a line in the finite projective plane

$$\Rightarrow \quad e_P(e_Q e_R) = -(e_P e_Q)e_R \quad \text{if PQR is not a line}$$

Conjugation: $x = \xi_0 + \sum \xi_i e_i \Rightarrow \bar{x} = \xi_0 - \sum \xi_i e_i$

$$x\bar{x} = |x|^2 = \xi_0^2 + \sum \xi_i^2 \Rightarrow \text{division } x^{-1} = \frac{\bar{x}}{|x|^2}$$

$$\overline{xy} = \bar{y}\,\bar{x}$$

2.jpg

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Composition: $|xy|^2 = |x|^2 \cdot |y|^2$

Alternative law: $[x, y, z] = x(yz) - (xy)z$

is totally antisymmetric

$$\equiv \quad x(xy) = x^2 y, \quad (yx)x = yx^2$$

THE CAYLEY-DICKSON PROCESS

Let $\mathbb{K}$ be a real conjugation algebra, and define a multiplication on $\mathbb{K}^2_\alpha = \mathbb{K} + h\mathbb{K}$ $(\alpha = \pm 1)$ by

$$h^2 = \alpha \in \mathbb{R} \subseteq \mathbb{K}$$

$$x(hy) = h(\overline{x}y), \quad (hx)y = h(yx), \quad (hx)(hy) = \alpha y\overline{x}$$

Then $\mathbb{K}^2_\alpha$ is a conjugation algebra, and

$$\begin{aligned}
\mathbb{K}^2_\alpha \text{ is commutative } &\iff \mathbb{K} = \mathbb{R}; \\
\mathbb{K}^2_\alpha \text{ is associative } &\iff \mathbb{K} \text{ is commutative}; \\
\mathbb{K}^2_\alpha \text{ is alternative } &\iff \mathbb{K} \text{ is associative} \\
&\implies \mathbb{K}^2_\alpha \text{ is a composition algebra} \\
&\implies \mathbb{K}^2_\alpha \text{ is a division algebra} \\
&\qquad \text{if } \alpha = -1 \text{ and } \mathbb{K} \text{ is a division algebra.}
\end{aligned}$$

DERIVATION ALGEBRAS OF $\mathbb{K}$

$\mathrm{Der}\,\mathbb{K}$ = set of $D : \mathbb{K} \to \mathbb{K}$ satisfying

$$D(xy) = (Dx)y + x(Dy)$$

= Lie algebra of $\mathrm{Aut}\,\mathbb{K}$, the set of $\Phi : \mathbb{K} \to \mathbb{K}$

satisfying $\quad \Phi(xy) = (\Phi x)(\Phi y)$

1.jpg

DERIVATION ALGEBRAS

$\mathrm{Der}\ \mathbb{R} = 0$	$\mathrm{Aut}\ \mathbb{R} = \{1\}$
$\mathrm{Der}\ \mathbb{C} = 0$	$\mathrm{Aut}\ \mathbb{C} = \{1, \bar{\ }\}$
$\mathrm{Der}\ \mathbb{H} = \mathfrak{sp}(1)$ $= \{a \in \mathbb{H} : \bar{a} = -a\}$ Inner: $Dx = ax - xa$	$\mathrm{Aut}\ \mathbb{H} = Sp(1)$ $\simeq \{u \in \mathbb{H} : u\bar{u} = 1\}$ $\Phi x = uxu^{-1}$
$\mathrm{Der}\ \mathbb{O} = \mathfrak{g}_2$	$\mathrm{Aut}\ \mathbb{O} = \text{"}G_2\text{"}$

THE MAGIC SQUARE

Let $\mathbb{K}$ be a real composition algebra, $\mathbb{J}$ a real Jordan algebra with identity and compatible inner product. Tits defined a bracket on

$$T(\mathbb{K}, \mathbb{J}) = \mathrm{Der}\mathbb{K} + \mathrm{Der}\mathbb{J} + \mathbb{K}' \otimes \mathbb{J}'$$

($A' =$ subspace of A orthogonal to 1) which yields a Lie algebra if *either* $\mathbb{K}$ is associative *or* $\mathbb{J}$ satisfies a certain cubic identity.

Taking $\mathbb{K} = \mathbb{K}_1$, $\mathbb{J} = H_3(\mathbb{K}_2)$ gives

$\mathbb{K}_1 \backslash \mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$sp(3)$	f_4
$\mathbb{C}$	$su(3)$	$su(3) \oplus su(3)$	$su(6)$	e_6
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	e_7
$\mathbb{O}$	f_4	e_6	e_7	e_8

Magic! 1. Exceptional Lie algebras

2. Symmetry

$\mathbb{J}=\mathbb{R}$ $\mathrm{Der}\,\mathbb{K}_1$	$\mathbb{K}_1 \backslash \mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
0	$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
0	$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)\oplus\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathfrak{su}(2)$	$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathfrak{g}_2$	$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Magic!

1. Exceptional Lie algebras
2. Symmetry

THE SYMMETRY OF THE MAGIC SQUARE

$$\mathcal{L}_3(\mathbb{K}_1, \mathbb{K}_2) = T\big(\mathbb{K}_1, H_3(\mathbb{K}_2)\big)$$

$$= \mathrm{Der}\,\mathbb{K}_1 + \mathrm{Der}\,H_3(\mathbb{K}_2) + \mathbb{K}_1' \otimes H_3'(\mathbb{K}_2)$$

Why is this symmetric between $\mathbb{K}_1$ and $\mathbb{K}_2$?

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N.B. $\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} + A_3'(\mathbb{K})$

A_3' = traceless antihermitian 3×3 matrices

i.e. $\operatorname{Der} H_3(\mathbb{K}) =$ "$\mathfrak{su}(3, \mathbb{K})$"

LIE ALGEBRAS ASSOCIATED WITH HERMITIAN MATRICES

$\mathrm{Der}\, H_n(\mathbb{K}) = \mathrm{ah}(n, \mathbb{K})$?

$A \longmapsto [X, A]$

$A \in H_n(\mathbb{K})$

$X \in \mathrm{ah}(n, \mathbb{K})$

Derivation: $[X, \{A, B\}] = \{[X, A], B\} + \{A, [X, B]\}$

Lie bracket: $[X, [Y, A]] - [Y, [X, A]] = [[X, Y], A]$

if $\mathbb{K}$ is associative

In fact, $\mathrm{Der}\, H_n(\mathbb{K}) = \mathrm{ah}(n, \mathbb{K}) - \mathbb{K}1 + \mathrm{Der}\, \mathbb{K}$

Also works for $n = 2, 3$ if $\mathbb{K} = \mathbb{O}$: $\mathrm{Der}\, H_3(\mathbb{K}) = \mathrm{Tri}\, \mathbb{K} \oplus 3\mathbb{K}$

of hermitian 2.jpg

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Theorem (Vinberg)

$$\mathcal{L}_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Der}\,\mathbb{K}_1 \oplus \mathrm{Der}\,\mathbb{K}_2 + A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$$

$$= \text{"}su(3, \mathbb{K}_1 \otimes \mathbb{K}_2)\text{"}$$

TRIALITY

Def: (Ramond) The Triality algebra of a composition algebra $\mathbb{K}$ is the Lie algebra

$$\mathrm{Tri}\,\mathbb{K} = \{ (D, E, F) \in \mathfrak{so}(\mathbb{K})^3 : D(xy) = (Ex)y + x(Fy) \}$$

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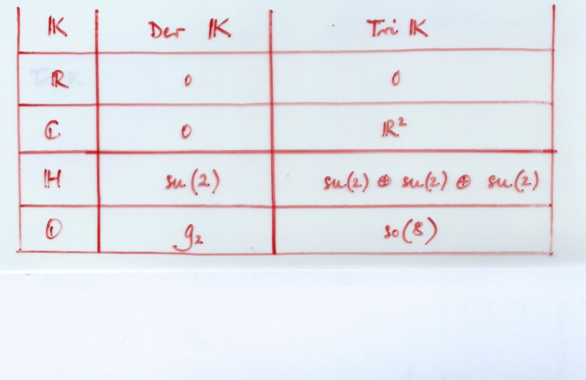
Principle of Triality

There is a 1:1 correspondence between $D \in \mathfrak{so}(8)$ & $(D, E, F) \in \mathrm{Tri}\,\mathbb{O}$, & the maps $D \mapsto E$, $D \mapsto F$ are inequivalent rep's of $\mathfrak{so}(8)$.



$\mathbb{K}$	$\mathrm{Der}\ \mathbb{K}$	$\mathrm{Tri}\ \mathbb{K}$
$\mathbb{R}$	0	0
$\mathbb{C}$	0	$\mathbb{R}^2$
$\mathbb{H}$	$\mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathbb{O}$	$\mathfrak{g}_2$	$\mathfrak{so}(8)$

$\mathbb{J} = \mathrm{diag}\, H_3$ $\mathrm{Tri}\, \mathbb{K}_1$	$\mathbb{J} = \mathbb{R}$ $\mathrm{Der}\, \mathbb{K}_1$	$\mathbb{K}_1 \backslash \mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
0	0	$\mathbb{R}$	$so(3)$	$su(3)$	$sp(3)$	f_4
$\mathbb{R}^2$	0	$\mathbb{C}$	$su(3)$	$su(3) \oplus su(3)$	$su(6)$	e_6
$su(2)+su(2)+su(2)$	$su(2)$	$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	e_7
$so(8)$	g_2	$\mathbb{O}$	f_4	e_6	e_7	e_8

square 4.jpg

$\mathbb{K}$	$\mathrm{Der}\ \mathbb{K}$	$\mathrm{Tri}\ \mathbb{K}$
$\mathbb{R}$	0	0
$\mathbb{C}$	0	$\mathbb{R}^2$
$\mathbb{H}$	$\mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathbb{O}$	$\mathfrak{g}_2$	$\mathfrak{so}(8)$

$$\mathrm{Der}\, H_3(\mathbb{K}) = \mathrm{Tri}\,\mathbb{K} + 3\mathbb{K}$$

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Tri}\,\mathbb{K}_1 \oplus \mathrm{Tri}\,\mathbb{K}_2 + 3\mathbb{K}_1 \otimes \mathbb{K}_2$$

INTRINSIC CLASSIFICATION THEOREM

Theorem (Allison)
Let $\mathfrak{g}$ be a simple Lie algebra containing a subalgebra $\mathfrak{su}(2)$ whose adjoint action on $\mathfrak{g}$ decomposes it into 1, 3 and 5-dimensional submodules. Then

$$\mathfrak{g} = \mathrm{Tri}(\mathcal{A}) + 3\mathcal{A}$$

for some structurable algebra $\mathcal{A}$.

Structurable algebras have been classified and include tensor products $\mathbb{K}_1 \otimes \mathbb{K}_2$ of composition algebras.

THE ROWS OF THE MAGIC SQUARE

Matrix models & Freudenthal geometries

Taking $\mathbb{K}_1$ to be a split comp^{n} algebra gives non-compact Lie algebras

$\mathbb{K}_1 \backslash \mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$su(3,\mathbb{H})$	$f_4(-52) = su(3,\mathbb{O})$
$\tilde{\mathbb{C}}$	$sl(3,\mathbb{R})$	$sl(3,\mathbb{C})$	$su^*(6) = sl(3,\mathbb{H})$	$e_6(-26) = sl(3,\mathbb{O})$
$\tilde{\mathbb{H}}$	$sp(6,\mathbb{R})$	$su(3,3) \simeq sp(6,\mathbb{C})$	$so^*(12) = sp(6,\mathbb{H})$	$e_7(-25) = sp(6,\mathbb{O})$
$\tilde{\mathbb{O}}$	$f_4(4)$	$e_6(2)$	$e_7(-5)$	$e_8(-24)$

magic square.jpg

THE FIRST ROW

$$
\begin{aligned}
T(\mathbb{R}, H_3(\mathbb{K})) &= \mathrm{Der} H_3(\mathbb{K}) \\
&= A_3(\mathbb{K}) - \mathbb{K} + \mathrm{Der}\mathbb{K} \\
&= \mathfrak{su}(3, \mathbb{K})
\end{aligned}
$$

LIE ALGEBRAS ASSOCIATED WITH HERMITIAN MATRICES

$\mathrm{Der}\, H_n(\mathbb{K}) = ah(n, \mathbb{K})$?

$A \longmapsto [X, A]$ $\quad A \in H_n(\mathbb{K})$

$\quad X \in ah(n, \mathbb{K})$

Derivation: $[X, \{A, B\}] = \{[X, A], B\} + \{A, [X, B]\}$

Lie bracket: $[X, [Y, A]] - [Y, [X, A]] = [[X, Y], A]$

if $\mathbb{K}$ is associative

In fact, $\mathrm{Der}\, H_n(\mathbb{K}) = ah(n, \mathbb{K}) - \mathbb{K}1 + \mathrm{Der}\, \mathbb{K}$

Also works for $n = 2, 3$ if $\mathbb{K} = \mathbb{O}$: $\mathrm{Der}\, H_3(\mathbb{K}) = \mathrm{Tri}\, \mathbb{K} \oplus 3\mathbb{K}$

THE SECOND ROW

$$\begin{aligned} T(\widetilde{\mathbb{C}}, \mathbb{J}) &= \mathrm{Der}\mathbb{J} + \tilde{i}\mathbb{J}' \\ &= \mathrm{Str'}\mathbb{J} \end{aligned}$$

For $\mathbb{J} = H_3(\mathbb{K})$,

$$\mathrm{Str}'\,\mathbb{J} = \mathfrak{sl}(n, \mathbb{K})$$

$$\mathrm{Str}'H_n(\mathbb{K}) = sl(n,\mathbb{K}) \quad ?$$

$$A \mapsto \{B, A\} \qquad A, B \in H_n(\mathbb{K})$$

$$\{C,\{B,A\}\} - \{B,\{C,A\}\} = [[B,C],A]$$

so $\mathrm{Str}\,H_n(\mathbb{K}) = H_n(\mathbb{K}) \oplus ah(n,\mathbb{K}) = gl(n,\mathbb{K})$

$$A \mapsto XA + AX^{\dagger} \qquad A \in H_n(\mathbb{K}),\ X \in gl(n,\mathbb{K})$$

In fact $\mathrm{Str}'H_3(\mathbb{K}) = sl(3,\mathbb{K}) + \mathrm{Der}\,\mathbb{K}$

MAGIC SQUARE GEOMETRIES

(Freudenthal)

Group	Geometry	Elements	Relations
SU(3)	elliptic	points	polarity
SL(3)	projective plane	points lines	incident
Sp(6)	5-dimensional symplectic	points lines planes	joined interwoven
Exceptional	metasymplectic	points lines planes symplecta	joined interwoven hinged

MAKING THE SPLIT

Suppose $\mathbb{K}_1$ and $\mathbb{K}_2$ are positive definite composition algebras, $\widetilde{\mathbb{K}}$ the split form of $\mathbb{K}$. Then $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$ and $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$ contain non-compact Lie algebras. These can be identified by their maximal compact subalgebras.

The compact magic square $L_3(\mathbb{K}_1, \mathbb{K}_2)$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	F_4
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	E_6
$\mathbb{H}$	$\mathfrak{sq}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

.

Maximal compact subalgebras of $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	F_4
$\widetilde{\mathbb{C}}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	F_4
$\widetilde{\mathbb{H}}$	$\mathfrak{su}(3) \oplus \mathfrak{so}(2)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{so}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2)$	$E_6 \oplus \mathfrak{so}($
$\widetilde{\mathbb{O}}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$E_7 \oplus \mathfrak{so}($

Maximal compact subalgebras of $L_3(\widetilde{\mathbb{K}}_1, \mathbb{K}_2)$

Theorem Barton and AS
The maximal compact subalgebra of the non-compact magic square algebra $L_3(\widetilde{\mathbb{K}}_1 \otimes \mathbb{K}_2)$ is $L_3(\mathbb{F}_1 \otimes \mathbb{K}_2) \dot{+} \mathbb{F}_1'$, where $\mathbb{F}_1$ is the division algebra preceding $\mathbb{K}_1$ in the Cayley-Dickson process.

Maximal compact subalgebras of $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$\mathbb{C}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(6)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{su}(3)$	$\mathfrak{so}(6)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

.

containing

	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\mathfrak{so}(4)$	$\mathfrak{su}(4)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{su}(4)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(4)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

.

containing

	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\mathfrak{so}(4)$	$\mathfrak{su}(4)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{su}(4)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(4)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

.

which are the compact forms of

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(4)$	$\mathfrak{su}(4)$	$\mathfrak{sq}(4)$
$\mathbb{C}$	$\mathfrak{sl}(4, \mathbb{R})$	$\mathfrak{sl}(4, \mathbb{C})$	$\mathfrak{sl}(4, \mathbb{H})$
$\mathbb{H}$	$\mathfrak{sp}(8, \mathbb{R})$	$\mathfrak{sp}(8, \mathbb{C}) \cong \mathfrak{su}(4, 4)$	$\mathfrak{sp}(8, \mathbb{H})$

.

Theorem The maximal compact subalgebra of $L_3(\widetilde{\mathbb{K}}_1, \widetilde{\mathbb{K}}_2)$ is $L_4(\mathbb{F}_1, \widetilde{\mathbb{F}}_2)$ where $\mathbb{F}_i$ is the division algebra preceding $\mathbb{K}_i$ in the Cayley-Dickson process.

THE 2×2 MAGIC SQUARE

Jordan algebra $\quad H_2(\mathbb{K}) \cong V \oplus \mathbb{R} \subset \mathrm{Cliff}^+(V)$

$$V = \mathbb{K} \oplus \mathbb{R}$$

$$\Longrightarrow \quad \mathrm{Der}\, H_2(\mathbb{K}) = \mathfrak{su}(2, \mathbb{K}) \cong \mathfrak{so}(\mathbb{K} \oplus \mathbb{R})$$

$$\mathfrak{su}(2) \cong \mathfrak{so}(3)$$

$$\mathfrak{sp}(2) \cong \mathfrak{so}(5)$$

$$\mathfrak{su}(2, \mathbb{O}) \cong \mathfrak{so}(9)$$

magic square 1.jpg

$SL(2,\mathbb{K})$ acts on $X = \begin{pmatrix} \alpha & z \\ \bar{z} & \beta \end{pmatrix} \in H_2(\mathbb{K})$

$X \mapsto A X A^\dagger$ preserving $\det X = \alpha\beta - |z|^2$

$$\Rightarrow \quad \mathfrak{sl}(2,\mathbb{K}) \cong \mathfrak{so}(\mathbb{K} \oplus \mathbb{R}, \mathbb{R})$$

$$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$$

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$$

$$\mathfrak{su}^*(4) \cong \mathfrak{sl}(2,\mathbb{H}) \cong \mathfrak{so}(5,1)$$

$$\mathfrak{sl}(2,\mathbb{O}) \cong \mathfrak{so}(9,1)$$

magic square 2.jpg

Symplectic transf's of $\mathbb{K}^4$ ~ Möbius transf's of $H_2(\mathbb{K})$

$$\Rightarrow \quad \mathfrak{sp}(4, \mathbb{K}) \cong \mathfrak{so}(\nu+2, 2)$$

$$\mathfrak{sp}(4, \mathbb{R}) \cong \mathfrak{so}(3, 2)$$

$$\mathfrak{su}(2,2) = \mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(4, 2)$$

$$\mathfrak{so}^*(8) = \mathfrak{sp}(4, \mathbb{H}) \cong \mathfrak{so}(6, 2)$$

$$\mathfrak{sp}(4, \mathbb{O}) \cong \mathfrak{so}(10, 2)$$

DEFINING THE 2×2 MAGIC SQUARE

Tits's construction $T(\mathbb{K}_1, H_2(\mathbb{K}))$ gives a Lie algebra only if $\mathbb{K}_1$ is associative. But in that case

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\mathbb{K}_1) + \mathrm{Der} H_2(\mathbb{K}_2) + \mathbb{K}_1' \otimes H_2'(\mathbb{K}_2)$$

which is also a Lie algebra if $\mathbb{K}_1 = \mathbb{O}$. Then

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\mathbb{K}_1 \oplus \mathbb{K}_2).$$

THE SQUARE OF ISOMORPHISMS

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\widetilde{\mathbb{O}}$
$\mathrm{Der}\, H_2(\mathbb{K}) \cong L_2(\mathbb{R}, \mathbb{K})$	$\mathfrak{so}(2)$	$\mathfrak{su}(2)$	$\mathfrak{sq}(2)$	$\mathfrak{so}(9)$
$\mathrm{Str}\, H_2(\mathbb{K}) \cong L_2(\widetilde{\mathbb{C}}, \mathbb{K})$	$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{H})$	$\mathfrak{sl}(2, \mathbb{O})$
$\mathrm{Con}\, H_2(\mathbb{K}) \cong L_2(\widetilde{\mathbb{H}}, \mathbb{K})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{su}(2, 2)$	$\mathfrak{sp}(4, \mathbb{H})$	$\mathfrak{sp}(4, \mathbb{O})$
$L_2(\widetilde{\mathbb{O}}, \mathbb{K})$	$\mathfrak{so}(5, 4)$	$\mathfrak{so}(6, 4)$	$\mathfrak{so}(8, 4)$	$\mathfrak{so}(12, 4)$

THE EXCEPTIONAL SERIES

Observation (Vogel) Every simple Lie algebra can be associated with a set of six points $(\alpha, \beta, \gamma) \in \mathbb{Q}\mathbb{P}^2$ (related by permutations)

s.t.

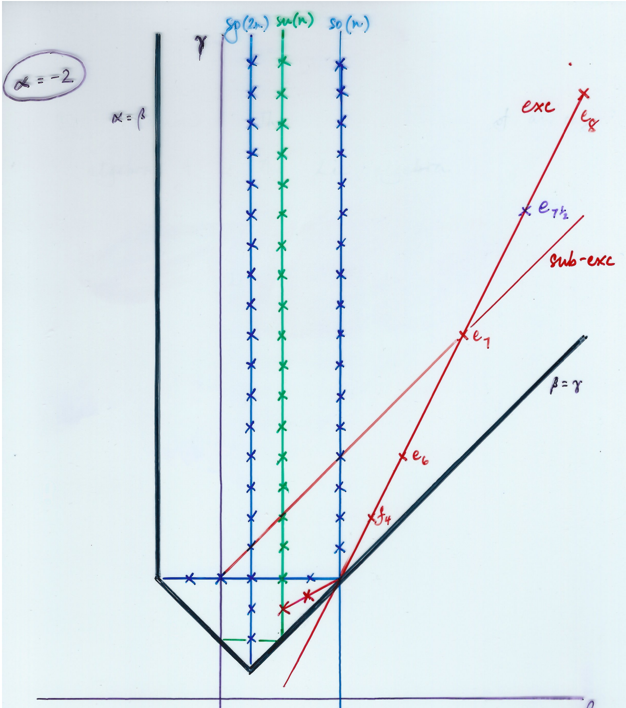
$$\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g} \oplus X_2 \qquad \text{(antisym.)}$$

$$\oplus\ X_0 \oplus Y_2 \oplus Y_2' \oplus Y_2''$$

with $$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} \qquad (t = \alpha + \beta + \gamma)$$

$$\dim X_2 = -\frac{(\alpha-2t)(\beta-2t)(\gamma-2t)(\alpha+t)(\beta+t)(\gamma+t)}{\alpha^2\beta^2\gamma^2}$$

& similar formulae for Y_2, Y_2', Y_2''.



α = −2
sp(2n)
su(n)
so(n)
γ
α = β
exc
e8
e7½
sub-exc
e7
β = γ
e6
f4

Conjecture (Deligne)

The Lie algebras $\mathfrak{g}$ in the exceptional series $(\alpha, \beta, \gamma) = (\lambda, 1-\lambda, 2)$ have similar decompositions of $\otimes^k \mathfrak{g}$ for all k, & the dimensions & values of the quadratic Casimir on the irreducible components are given by products $\prod_i (a_i + \lambda b_i)^{\pm 1}$.

$(a_i, b_i \in \mathbb{Q})$

Verified by Cohen & de Man for $k = 3, 4$

Proved analytically by Landsberg & Manivel, $k = 2, 3$

Investigated by Macfarlane & Pfeiffer for $k = 5$

– quadratic factors, negative dimensions

conjecture.jpg

THE SEXTONIONS AND $E_{7\frac{1}{2}}$

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}^{\perp}$$

Under left mult^{n} by $\mathbb{H}^{\times} \cong \mathbb{R} \times SU(2)$,

$\mathbb{C}\mathbb{H}^{\perp} = U_1 \oplus U_2$ $\qquad$ $\dim U_1 = \dim U_2 = 2$

The <u>sextonions</u> are the complex 6-dim. algebra

$$\mathbb{S} = \mathbb{C} \otimes \left(\mathbb{H} + U_1\right)$$

This adds a frieze to the magic square

$$L_3(\mathbb{S}, \mathbb{K}) = L_3(\mathbb{H}, \mathbb{K}) \rtimes H(\mathbb{K})$$

$H(\mathbb{K})$ = Heisenberg algebra of dimension

14,	20,	32,	32 × 44,	56
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{S}$	$\mathbb{O}$

so $$E_{7\frac{1}{2}} = E_7 \rtimes H_{56}$$

INTERIOR DECORATION OF THE MAGIC SQUARE

$\mathbb{K}_1 \backslash \mathbb{K}_2$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{S}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$sp(3)$	$sp(3)+h_{14}$	f_4
$\mathbb{C}$	$su(3)$	$su(3)\oplus su(3)$	$su(6)$	$su(6)+h_{20}$	e_6
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	$so(12)+h_{32}$	e_7
$\mathbb{S}$	$sp(3)+h_{14}$	$su(6)+h_{20}$	$so(12)+h_{32}$	$so(12)+h_{32}+h_{44}$	e_7+h_{56}
$\mathbb{O}$	f_4	e_6	e_7	e_7+h_{56}	e_8

magic square.jpg

THE EXCEPTIONAL SERIES

Observation (Vogel) Every simple Lie algebra can be associated with a set of six points $(\alpha, \beta, \gamma) \in \mathbb{QP}^2$ (related by permutations) s.t.

$$\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g} \oplus X_2 \quad \text{(antisym.)}$$

$$\oplus X_0 \oplus Y_2 \oplus Y_2' \oplus Y_2''$$

with $\dim \mathfrak{g} = \dfrac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$ $\quad (t = \alpha + \beta + \gamma)$

$$\dim X_2 = -\frac{(\alpha-2t)(\beta-2t)(\gamma-2t)(\alpha+t)(\beta+t)(\gamma+t)}{\alpha^2\beta^2\gamma^2}$$

& similar formulae for Y_2, Y_2', Y_2''.

UNIVERSAL DIMENSION FORMULA

Landsberg & Manivel 2004
math. RT/0401296

The k'th symmetric power of $\mathfrak{g}$ contains an irreducible rep^{n} Y_k (highest weight $k\alpha_0$) with dimension

$$\frac{(\beta+\gamma-3+2k)\binom{\beta+\frac{\gamma}{2}-3+k}{k}\binom{\gamma+\frac{\beta}{2}-3+k}{k}\binom{\beta+\gamma-4+k}{k}}{(\beta+\gamma-3)\binom{\frac{\beta}{2}+k-1}{k}\binom{\frac{\gamma}{2}+k-1}{k}}$$

dimension formula.jpg

TENSOR CALCULUS

1968

a_n adjoint tensors:

$$\lambda_i \lambda_j = \frac{2}{n}\delta_{ij} + (d_{ijk} + i f_{ijk})\lambda_k$$

Identities

$$f_{ijk} f_{mnk} = \frac{2}{n}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) + d_{imk} d_{jnk} - d_{ink} d_{jmk} \quad \text{etc.}$$

Macfarlane, AS, Weisz

TENSOR CALCULUS

1968

a_n adjoint tensors:

$$\lambda_i \lambda_j = \frac{2}{n} \delta_{ij} + (d_{ijk} + i f_{ijk}) \lambda_k$$

Identities

$$f_{ijk} f_{mnk} = \frac{2}{n}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})$$
$$+ d_{imk} d_{jnk} - d_{ink} d_{jmk} \quad \text{etc.}$$

Macfarlane, AS, Weisz

2002

$$\underset{ijpq}{S} \; c_{imn} c_{pnr} c_{qrs} c_{jsm} = \frac{5}{2(D+2)} \delta_{(ij} \delta_{pq)}$$

$$D = \dim \underset{\sim}{g}, \quad \underset{\sim}{g} = \underset{\sim}{a}_2, \underset{\sim}{g}_2, \underset{\sim}{f}_4, \underset{\sim}{e}_6, \underset{\sim}{e}_7, \underset{\sim}{e}_8$$

! ! ! !

Macfarlane & Pfeiffer