Partial transversals in a class of latin squares

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  - Partial transversals.
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    - Constructions of transversals.
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    - Results so far.

Latin squares.

Definition

A latin square of side n is an $n \times n$ matrix in which each symbol from an n-element set appears exactly once in each row and each column.

Latin squares.

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Example

A latin square of order 9.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\ 3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\ 4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 1 \\ 6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\ 7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\ 8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\ 9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \end{pmatrix}$$

Partial transversals.

Definition

A partial transversal of length m in a latin square L is a set T of m cells,

- at most one from each row,
- at most one for each column,
- no symbol of L appearing more than once in T.

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- at most one for each column,
- no symbol of L appearing more than once in T.

Definition

A partial transversal is maximal if it cannot be extended to a partial transversal of greater length.

A partial transversal of length 5 in a latin square of side 9.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\ 3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\ 4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 1 \\ 6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\ 7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\ 8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\ 9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \end{pmatrix}$$

A partial transversal of length 5 in a latin square of side 9.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 7 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot \\ 5 & 6 & 7 & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 7 & 8 & 9 & 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 8 & 9 & 5 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 \end{pmatrix}$$

This partial transversal is maximal.

A partial transversal of length 7 in a latin square of side 9.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 \\ 3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 \\ 4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 1 \\ 6 & 7 & 8 & 9 & 1 & 5 & 2 & 3 & 4 \\ 7 & 8 & 9 & 5 & 4 & 1 & 6 & 2 & 3 \\ 8 & 9 & 5 & 6 & 3 & 4 & 1 & 7 & 2 \\ 9 & 5 & 6 & 7 & 2 & 3 & 4 & 1 & 8 \end{pmatrix}$$

A partial transversal of length 7 in a latin square of side 9.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 8 & \cdot & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 9 & \cdot & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot \\ \cdot & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 \end{pmatrix}$$

This partial transversal is maximal.

Notes on partial transversals.

L a latin square of side n.

- A partial transversal of length n is a transversal.

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Theorem

If a latin square of side n has a maximal partial transversal of lenght m, then

$$\left\lceil \frac{n}{2} \right\rceil \leq m \leq n.$$

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Example

Our latin square of side 9 has maximal partial transversals of lengths 5, 6, 7, 8, and 9, i.e., all allowed lengths.

Notation.

M_n will denote the Cayley table of $\mathbb{Z}_n$, i.e., the latin square of side n and ijth entry

$$i + j \mod n,$$

$i, j = 0, \ldots, n-1$.

Bicyclic latin squares.

A latin square $L_{A,d_1,\dots,d_n}$ of side $2n+1$ is bicyclic if its symbol set is

$$\mathbb{Z}_n \cup \mathbb{Z}_{n+1} = \{0, 1, \dots, n-1\} \cup \{0, 1, \dots, n\},$$

and

$$L_{A,d_1,\dots,d_n} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where

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where

- A is isotopic to M_n,

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and

$$L_{A,d_1,\ldots,d_n} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

where

- A is isotopic to M_n,
- B is M_{n+1} with the last row removed,
- C is M_{n+1} with the last column removed, and
- D has the last row of B/last column of C on the main diagonal, and ijth entry $d_{j-i} \in \mathbb{Z}_n$, $i \neq j$.

Bicyclic latin squares.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

This is $L_{M_4,1,2,3,0}$.

Some basic facts.

Lemma

If n is even and a bicyclic latin square of side $2n+1$ has a maximal partial transversal of length m, then

$$n+1 \leq m \leq 2n+1.$$

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Lemma

If n is even, then M_n

- *has no transversals, and*
- *any cell can be the missing cell of a near transversal.*

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Lemma

If n is even, then M_n

- *has no transversals, and*
- *any cell can be the missing cell of a near transversal.*

Lemma

If n is odd, then M_n

- *has transversals, and*
- *any near transversal can be extended to a transversal.*

The case n even. Constructions of maximal partial transversals.

Lemma

If n is even, then any bicyclic latin square of side $2n + 1$ has a maximal partial transversal of length $n + 1$.

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Lemma

If n is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $n+1$.

The construction.

For

$$L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

pick a

- "transversal" of B, and

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Lemma

If n is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $n+1$.

The construction.

For

$$L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

pick a

- "transversal" of B, and
- one entry on the main diagonal of D.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\ 3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\ 4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3 \end{array}\right)$$

The black entries form a maximal partial transversal of length 5.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & \cdot & \cdot & \cdot & \cdot \\
1 & 2 & 3 & 0 & \cdot & 2 & \cdot & \cdot & \cdot \\
2 & 3 & 0 & 1 & \cdot & \cdot & 4 & \cdot & \cdot \\
3 & 0 & 1 & 2 & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 0 \\
\cdot & \cdot & \cdot & \cdot & 0 & \cdot & 1 & 2 & 3 \\
\cdot & \cdot & \cdot & \cdot & 3 & 0 & \cdot & 1 & 2 \\
\cdot & \cdot & \cdot & \cdot & 2 & 3 & 0 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

The black entries form a maximal partial transversal of length 5.

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Lemma

If n is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $2n$.

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If n is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $2n$.

The construction.

For

$$L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

pick a

- near transversal of A, and

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If n is even, then any bicyclic latin square of side $2n+1$ has a maximal partial transversal of length $2n$.

The construction.

For

$$L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

pick a

- near transversal of A, and
- the main diagonal of D.

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A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

The entries shown, less one 1 form a maximal partial transversal of length 8.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3
\end{array}\right)$$

The entries shown, less one 1, form a maximal partial transversal of length 8.

The case n even. Constructions of maximal partial transversals.

Lemma

If n is even and

$$1 \leq k < \frac{n}{2},$$

then $L_{M_n,1,\ldots,n-1,0}$ has a maximal partial transversal of length $n + 2k + 1$.

The case n even. Constructions of maximal partial transversals.

The construction.

First pick the black entries shown below.

$$\left(\begin{array}{ccc|ccc}
 & & & 0 & & \\
 & & & & \ddots & \\
 & & & & & 2(k-1) \\
\hline
2k & & & & & \\
 & \ddots & & & & \\
 & & 2(n-1) & & & \\
 & & & & & 2n
\end{array}\right)$$

Then choose blue entries.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal maximal partial transversal of length 7.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
\cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3
\end{array}\right)$$

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal maximal partial transversal of length 7.

The case n even. Constructions of maximal partial transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
\cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\
\color{red}{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\color{blue}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\color{blue}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \color{blue}{1} & \color{red}{2} & \color{blue}{3} & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3
\end{array}\right)$$

The entries shown form a partial transversal of length 5, which can be extended. The red and black entries form a maximal maximal partial transversal of length 7.

The case n even. Constructions of maximal partial transversals.

Lemma

If n is even and

$$1 \leq k < \frac{n}{2},$$

then $L_{M_n,1,\ldots,n-1,0}$ has a maximal partial transversal of length $n+2k$.

The case n even. Constructions of maximal partial transversals.

The construction.

First pick the black entries shown below.

$$\left(\begin{array}{ccc|ccc}
 & & & 0 & & \\
 & & & & \ddots & \\
 & & & & & 2(k-1) \\ \hline
 & & 2n & & & \\
2k & & & & & \\
 & \ddots & & & & \\
 & & 2(n-1) & & &
\end{array}\right)$$

Then choose blue entries.

The case n even. Constructions of transversals.

A question.

For

$$L_{A,d_1,\dots,d_n} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

a bicyclic latin square of side $2n+1$, n even, when can a near transversal of A be extended to a transversal of $L_{A,d_1,\dots,d_n}$?

The case n even. Constructions of transversals.

A question.

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$$L_{A,d_1,\ldots,d_n} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

a bicyclic latin square of side $2n+1$, n even, when can a near transversal of A be extended to a transversal of $L_{A,d_1,\ldots,d_n}$?

A possible extension.

Let T be a near transversal of A.

- Missing cell (i,j) with entry a.

The case n even. Constructions of transversals.

A question.

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$$L_{A,d_1,\ldots,d_n} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

a bicyclic latin square of side $2n+1$, n even, when can a near transversal of A be extended to a transversal of $L_{A,d_1,\ldots,d_n}$?

A possible extension.

Let T be a near transversal of A.

- Missing cell (i,j) with entry a.
- Missing symbol $-a \in \mathbb{Z}_n$.

The case n even. Constructions of transversals.

A possible extension.

- Pick cell (i, t) in B: entry $y = i + t \in \mathbb{Z}_{n+1}$.

The case n even. Constructions of transversals.

A possible extension.

- Pick cell (i, t) in B: entry $y = i + t \in \mathbb{Z}_{n+1}$.
- Pick cell (s, j) in C: entry $x = s + j \in \mathbb{Z}_{n+1}$.

The case n even. Constructions of transversals.

A possible extension.

$$
\left(\begin{array}{cccc|ccccc}
 & & \color{blue}{j} & & & t & & s & \\
 & & \vdots & & & \vdots & & \vdots & \\
\color{blue}{i} & \cdots & \color{green}{a} & \cdots & \cdots & y & \cdots & & \cdots \\
 & & \vdots & & & \vdots & & \vdots & \\
\hline
 & & \vdots & & \ddots & \vdots & & \vdots & \\
t & \cdots & & \cdots & \cdots & \color{green}{x} & \cdots & ? & \\
 & & \vdots & & & & \ddots & \vdots & \\
s & \cdots & x & \cdots & \cdots & & \cdots & \color{green}{y} & \\
 & & & & & & & & \ddots
\end{array}\right)
$$

The case n even. Constructions of transversals.

A possible extension.

We find that T can be extended to a transversal if and only if

The case n even. Constructions of transversals.

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We find that T can be extended to a transversal if and only if

- $i + j + 2 \equiv 0 \pmod{n+1}$, i.e., the missing cell of T is on the antidiagonal of A, and

The case n even. Constructions of transversals.

A possible extension.

We find that T can be extended to a transversal if and only if

- $i + j + 2 \equiv 0 \pmod{n+1}$, i.e., the missing cell of T is on the antidiagonal of A, and
- the "?" is $-a \in \mathbb{Z}_n$.

The case n even. Constructions of transversals.

A bicyclic latin square of side 9.

$$
\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)
$$

The entries shown form a transversal.

The case n even. Constructions of transversals.

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0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
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1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
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A bicyclic latin square of side 9.

$$
\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)
$$

The entries shown form a transversal.

The case n even. Constructions of transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0? \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

The entries shown form a transversal.

The case n even. Constructions of transversals.

A bicyclic latin square of side 9.

$$\left(\begin{array}{cccc|ccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 \\
2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 3 & 0 & 1 & 1 & 2 \\
3 & 4 & 0 & 1 & 2 & 3 & 0 & 2 & 1 \\
4 & 0 & 1 & 2 & 1 & 2 & 3 & 0 & 3
\end{array}\right)$$

The entries shown form a transversal.

The case n even. Constructions of transversals.

A bicyclic latin square of side 9.

$$
\left(\begin{array}{cccc|ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 \\
\cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot &
\end{array}\right)
$$

The entries shown form a transversal.

The case n even. The conclusion.

Theorem

If n is even, then there exists a bicyclic latin square of side $2n+1$ that has maximal partial transversals of all allowed lengths.

The case n odd. Results so far.

Theorem

If n is odd, then there exists a bicyclic latin square of side $2n+1$ that has maximal partial transversals of all allowed lengths except possibly $n+1$ and $2n$.