INVERSE SEMIQUANDLES

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Conjugation in Groups: Quandles

If we view conjugation in a group G as a pair of binary operations

$$x \triangleleft y = y^{-1}xy$$
$$x \triangleleft^{-1} y = yxy^{-1}$$

then we obtain a (by now familiar) algebraic structure $\mathrm{Conj}(G) = (G, \triangleleft, \triangleleft^{-1})$ known as a *quandle*:

$(x \triangleleft y) \triangleleft^{-1} y = x = (x \triangleleft^{-1} y) \triangleleft y$	right quasigroup
$x \triangleleft x = x$	idempotent
$(x \triangleleft z) \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft z$	right distributive

Sufficient Axioms

$$(x \triangleleft y) \triangleleft^{-1} y = x = (x \triangleleft^{-1} y) \triangleleft y \qquad \text{right quasigroup}$$

$$x \triangleleft x = x \qquad \text{idempotent}$$

$$(x \triangleleft z) \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft z \qquad \text{right distributive}$$

Theorem (Joyce 1982)

Any identity holding in $\mathrm{Conj}(G)$ for all groups G holds in all quandles.

In universal algebra jargon, the variety of algebras generated by all $\mathrm{Conj}(G)$ is precisely the variety of all quandles.

Inverse Semigroups

Let S be a semigroup and fix $a \in S$. An element $b \in S$ is said to be an *inverse* of a if $aba = a$ and $bab = b$.

S is an *inverse semigroup* if every element has a *unique* inverse.

Equivalently: every element has an inverse and the idempotents commute *with each other*

$$ee = e \quad \& \quad ff = f \qquad \Longrightarrow \qquad ef = fe\,.$$

Symmetric Inverse Semigroup

The fundamental example is the *symmetric inverse semigroup* on a set X:

$$I_X = \{\alpha : A \to B \mid A, B \subseteq X, \alpha \text{ bijective}\}$$

How partial transformations compose is best seen in a picture.

I_X contains the symmetric group S_X as a sub(semi)group.

Theorem (Wagner-Preston)

Every inverse semigroup embeds in some I_X.

Philosophy

Groups	Symmetry
Quasigroups	Approximate Symmetry†
Inverse Semigroups	Partial Symmetry

(† Talk to J.D.H. Smith if you want this philosophy fleshed out in detail.)

Axioms

Inverse semigroups can be viewed as a variety of (universal) algebras $(S, \cdot, ^{-1})$ defined by equational axioms, such as

$$
\begin{aligned}
(xy)z &= x(yz) \\
xx^{-1}x &= x \\
(x^{-1})^{-1} &= x \\
xx^{-1} \cdot y^{-1}y &= y^{-1}y \cdot xx^{-1}
\end{aligned}
$$

Conjugacy

There are many nonequivalent ways to define conjugacy in semigroups. In inverse semigroups, the "naive" definition

$$a \text{ is conjugate to } b \quad \text{iff} \quad g^{-1}ag = b \text{ and } gbg^{-1} = a \text{ for some } g \in S$$

is of some interest.

(Note that *both* equations are needed here.)

Conjugation Inverse Semiquandles

Let's try the same idea as before. In an inverse semigroup S, define

$$x \triangleleft y = y^{-1}xy$$
$$x \triangleleft^{-1} y = yxy^{-1}$$

Call $\mathrm{Conj}(S) = (S, \triangleleft, \triangleleft^{-1})$ a *conjugation inverse semiquandle*.

Any algebra $(S, \triangleleft, \triangleleft^{-1})$ in the variety generated by all $\mathrm{Conj}(S)$ will be called an *inverse semiquandle*. We'll redefine this notion by axioms once we figure out what those axioms are.

Remark on Terminology

I do not have a general definition of “semiquandle”.

Henrich & Nelson (2010) use “semiquandle” to mean something else and I don’t think inverse semiquandles fit their sense of the term.

But I like the name “inverse semiquandle”. It evokes where these things come from.

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But I like the name “inverse semiquandle”. It evokes where these things come from.

So for now, think of the status of “semiquandle” as like what “quantum group” used to be: it means whatever an author wants it to mean until things settle down.

Identities

Instead of the right quasigroup axioms, the two right multiplications in inverse semiquandles are inverses:

$$((x \lhd y) \lhd^{-1} y) \lhd y = x \lhd y$$
$$((x \lhd^{-1} y) \lhd y) \lhd^{-1} y = x \lhd^{-1} y$$

Instead of idempotence, we have these:

$$(x \lhd x) \lhd^{-1} x = x \lhd^{-1} x$$
$$(x \lhd^{-1} x) \lhd x = x \lhd x$$

Distributivity

The right distributive law does not hold, in general. Instead, we have these:

$$((z \triangleleft^{-1} x) \triangleleft y) \triangleleft x = z \triangleleft (y \triangleleft x)$$
$$((z \triangleleft^{-1} x) \triangleleft^{-1} y) \triangleleft x = z \triangleleft^{-1} (y \triangleleft x)$$
$$((z \triangleleft x) \triangleleft y) \triangleleft^{-1} x = z \triangleleft (y \triangleleft^{-1} x)$$
$$((z \triangleleft x) \triangleleft^{-1} y) \triangleleft^{-1} x = z \triangleleft^{-1} (y \triangleleft^{-1} x)$$

Stare at these for a moment and notice that in quandles, they are equivalent to the right distributive laws.

Right multiplication maps

Triangle notation is unwieldy, so introduce right multiplication maps:

$$y\rho_x^+ = y \lhd x$$
$$y\rho_x^- = y \lhd^{-1} x$$

Here are some of the identities we have so far.

ρ_x^+ and ρ_x^- are inverses:

$$\rho_x^+ \rho_x^- \rho_x^+ = \rho_x^+$$
$$\rho_x^- \rho_x^+ \rho_x^- = \rho_x^-$$

Replacements for distributivity:

$$\rho_x^- \rho_y^+ \rho_x^+ = \rho_{y \lhd x}^+$$
$$\rho_x^- \rho_y^- \rho_x^+ = \rho_{y \lhd x}^-$$
$$\rho_x^+ \rho_y^+ \rho_x^- = \rho_{y \lhd^{-1} x}^+$$
$$\rho_x^+ \rho_y^- \rho_x^- = \rho_{y \lhd^{-1} x}^-$$

More identities

In $\mathrm{Conj}(S)$, every map of the form

$$\alpha = \rho_{x_1}^{\epsilon_1}\rho_{x_2}^{\epsilon_2}\dots\rho_{x_k}^{\epsilon_k}\rho_{x_k}^{-\epsilon_k}\dots\rho_{x_2}^{-\epsilon_2}\rho_{x_1}^{-\epsilon_1}$$

(where $\epsilon_i = \pm$) is idempotent ($\alpha^2 = \alpha$) and such maps commute with each other.
It turns out that it is sufficient to assume this

$$\rho_y^{\epsilon}\rho_y^{-\epsilon}\cdot\rho_{x_1}^{\epsilon_1}\dots\rho_{x_k}^{\epsilon_k}\rho_{x_k}^{-\epsilon_k}\dots\rho_{x_1}^{-\epsilon_1} = \rho_{x_1}^{\epsilon_1}\dots\rho_{x_k}^{\epsilon_k}\rho_{x_k}^{-\epsilon_k}\dots\rho_{x_1}^{-\epsilon_1}\cdot\rho_y^{\epsilon}\rho_y^{-\epsilon}$$

for each $k > 0$. Note that for each k, this describes 2^{k+1} identities.

Inverse semiquandles

$(Q, \triangleleft, \triangleleft^{-1})$ is an *inverse semiquandle* if it satisfies the following axioms

$$(x \triangleleft x) \triangleleft^{-1} x = x \triangleleft^{-1} x$$
$$(x \triangleleft^{-1} x) \triangleleft x = x \triangleleft x$$
$$\rho_x^+ \rho_x^- \rho_x^+ = \rho_x^+$$
$$\rho_x^- \rho_x^+ \rho_x^- = \rho_x^-$$
$$\rho_x^- \rho_y^+ \rho_x^+ = \rho_{y \triangleleft x}^+$$
$$\rho_x^- \rho_y^- \rho_x^+ = \rho_{y \triangleleft x}^-$$
$$\rho_x^+ \rho_y^+ \rho_x^- = \rho_{y \triangleleft^{-1} x}^+$$
$$\rho_x^+ \rho_y^- \rho_x^- = \rho_{y \triangleleft^{-1} x}^-$$
$$\rho_y^{\epsilon} \rho_y^{-\epsilon} \cdot \rho_{x_1}^{\epsilon_1} \ldots \rho_{x_k}^{\epsilon_k} \rho_{x_k}^{-\epsilon_k} \ldots \rho_{x_1}^{-\epsilon_1} = \rho_{x_1}^{\epsilon_1} \ldots \rho_{x_k}^{\epsilon_k} \rho_{x_k}^{-\epsilon_k} \ldots \rho_{x_1}^{-\epsilon_1} \cdot \rho_y^{\epsilon} \rho_y^{-\epsilon}$$

Dependencies?

There are some known dependencies in the axioms. The main one is this:

In the infinite sequence of identities, for each $k > 0$, the corresponding set of 2^{k+1} identities implies (in the presence of the other axioms) all the identities for smaller k.

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Conjecture

The variety of inverse semiquandles is not finitely based.

That is, I don't think one can axiomatize them with finitely many identities.
Nor do I think one can do this with finitely many variables.

Results I

Theorem

Any identity holding in $\mathrm{Conj}(S)$ *for all inverse semigroups* S *holds in all inverse semiquandles.*

Idea of proof: For a set X, let F be the free inverse semigroup on X and let Q be the set of all conjugates in F of elements of X. Show that Q is the free inverse semiquandle on X.

Results II

Theorem

Let Q be an inverse semiquandle. Then

$$R = \langle \rho_x^{\pm} \mid x \in Q \rangle$$

is an inverse semigroup, and R embeds in I_Q.

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Define the *associated inverse semigroup* of an inverse semiquandle Q:

$$\mathrm{As}(Q) = \langle Q \mid a^{-1}ba = b \triangleleft a, aba^{-1} = b \triangleleft^{-1} a \rangle .$$

Theorem

The map $Q \mapsto \mathrm{As}(Q)$ is a functor from the category of inverse semiquandles to the category of inverse semigroups, left adjoint to the functor $S \mapsto \mathrm{Conj}(S)$.

Clifford semiquandles

A *Clifford semigroup* is an inverse semigroup in which the idempotents commute with everything. These are precisely semilattices of groups. If we play the same game with Clifford semigroups, we get *Clifford semiquandles*:

$$x \lhd x = x$$
$$(x \lhd y) \lhd^{-1} y = (x \lhd^{-1} y) \lhd y$$
$$(x \lhd y) \lhd (z \lhd y) = (x \lhd z) \lhd y$$
$$((x \lhd^{-1} y) \lhd z) \lhd y = x \lhd (z \lhd y)$$
$$((x \lhd y) \lhd z) \lhd^{-1} y = x \lhd (z \lhd^{-1} y)$$
$$(((x \lhd^{-1} y) \lhd y) \lhd z) \lhd^{-1} z = (((x \lhd z) \lhd^{-1} z) \lhd^{-1} y) \lhd y$$

This is an independent set of identities. The same theorems hold in this setting.

Finale

Could inverse semiquandles be of any interest in knots, links, etc. if there were suitable cohomological invariants?
I would expect them to arise in situations where there are partial bijections but not full permutations.

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Thanks for your attention!