Higher Homotopy

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Example: $\mathbf{Qtp}(\mathbb{O}^*, \mathbb{O}^*)^* = \mathsf{Spin}_8(\mathbb{R})$

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Faithful **lowering functor** $\Lambda_3 : \mathbf{Qhh} \to \mathbf{Qtp}; (f, g, h; m) \mapsto (f, g, h)$

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iff $(A, +, 0)$ is Boolean.

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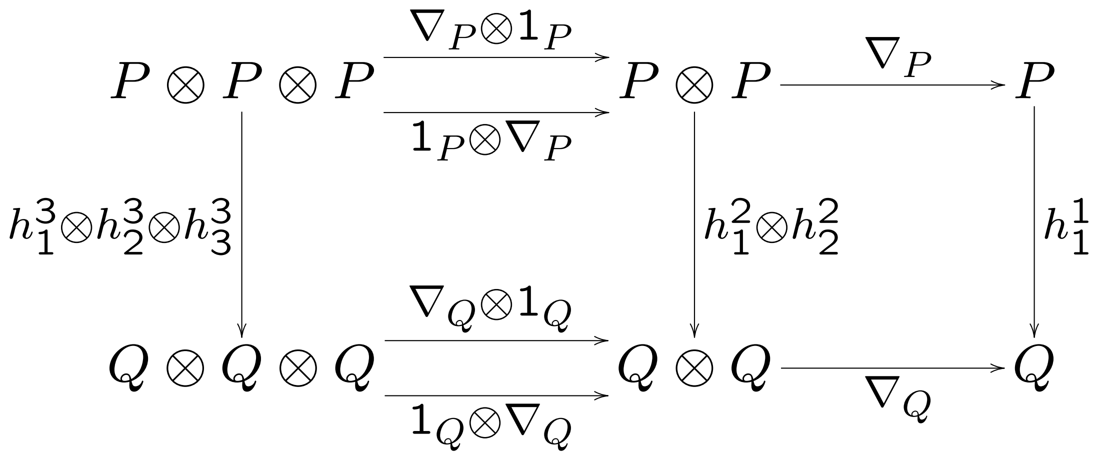

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P
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Then the quadruple $(h_1^2, h_2^2, h_1^1; h_2^3)\colon P \to Q$ is a higher homotopy iff the upper and lower left hand squares commute.

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with isomorphism $m\colon (P, *, e) \to (P, \cdot, 1); x \mapsto (e^f)^{-1} x (e^g)^{-1}$.

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Fact: Isotopes of loops are not necessarily loops.

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Fact: Higher isotopes of loops are loops.

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[Extension condition says $u\backslash u = d^f\backslash d^f = d^m = d^g/d^g = v/v$]

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Proof: A nonempty quasigroup Q is a loop iff $x\backslash x = y/y$ holds. $\square$

Example:

For a loop $(G, \cdot, /, \backslash, 1)$ with set I of involutions,
have $\mathbf{Qhh}\Big(\top, (G, /)\Big)\Lambda_3$ as the subset $G \times (I \cup \{1\})$
of the 3-net $\mathbf{Qtp}\Big(\top, (G, /)\Big)$ of $(G, /)$.

References

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Thank you for your attention!