QUASIGROUP FUNCTIONAL EQUATIONS OF DISTRIBUTIVITY

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Together with algebraic methods (quasigroup algebras and their varieties), combinatorial methods (Latin squares, cubes and hypercubes), geometric methods (nets) of investigation of quasigroups there exists a *functional method*. According to this method, quasigroups are considered as invertible functions which are defined on the same carrier. These functions form some algebras (groups, position algebras,. . .) or are some part of algebras (iterative algebras, bi-unary semigroups, Menger algebras,. . .). Theorems on solutions of functional equations are the most effective applicable results of this method.
In my talk, I will try to show this effectiveness on examples of application of the functional equation of generalized distributivity.

Corollaries from func. equ. of gener. associativity:

A quasigroup $(Q; \cdot)$ is isotopic to a group if it satisfies an identity:

1. $\dots(\dots x \dots y \dots) \dots z \dots = \dots x \dots (\dots y \dots z \dots) \dots$ (V.D. Belousov, 1987);
2. which is not Gemini (quadratic identity is gemini, if it is an identity in a free TS-loop) (A. Krapež, M. A. Taylor 1995);
3. which has a subterm $(x \overset{\sigma}{\cdot} y) \overset{\tau}{\cdot} (y \overset{\nu}{\cdot} z)$ for some quadratic variables x, y, z and some parastrophes $\overset{\sigma}{\cdot}, \overset{\tau}{\cdot}, \overset{\nu}{\cdot}$ of $(\cdot)$ (F. Sokhatsky);
4. $(x^n \cdot y^k) \cdot z = (z^p \cdot y) \cdot x$ (F. Sokhatsky).

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Corollaries from Pexider func. equation:

Proposition 2.(F.Sokhatsky,2016)

1. Isotopic inverse property loops are pseudo-isomorphic.
2. Isotopic commutative inverse property loops are isomorphic.

Definition

Two loops $(A;\cdot)$ and $(B;+)$ are said to be left pseudo-isomorphic, if there exists an element $a \in B$ and a mapping $\theta : A \to B$ such that $a + \theta(x \cdot y) = (a + \theta x) + \theta y$ for all $x, y \in A$.

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Invertibility

Left and right multiplications (Mann superpositions)

Let Q be an arbitrary set and $\Omega_2(Q)$ be the set of all binary operations defined on Q. One can define two associative superpositions on $\Omega_2(Q)$:

$$\left(f \underset{\ell}{\oplus} g\right)(x,y) := f(g(x,y),y), \qquad \left(f \underset{r}{\oplus} g\right)(x,y) := f(x,g(x,y))$$

which are called *left* and *right multiplications* respectively.

Two symmetric monoids

$(\Omega_2(Q); \underset{\ell}{\oplus}, e_\ell)$ and $(\Omega_2(Q); \underset{r}{\oplus}, e_r)$ are called *left* and *right symmetric monoids of binary operations*, where e_ℓ and e_r denote the left and right selectors: $e_\ell(x,y) := x$, $e_r(x,y) := y$.

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Invertibility of binary operations

A binary operation f is called:

- *left invertible*, if it is invertible in the left symmetric monoid and its inverse element is denoted by ${}^{\ell}f$;
- *right invertible*, if it is invertible in the right symmetric monoid and its inverse element is denoted by ${}^{r}f$;
- *invertible*, if it is both left and right invertible.

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Therefore, the algebra $(Q; f, {}^{\ell}f, {}^{r}f)$ is a *quasigroup*, if f is an invertible function and ${}^{\ell}f$, ${}^{r}f$ are its left and right inverses. f is also called a *quasigroup operation*.

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Parastrophe

σ-parastrophe of an arbitrary operation $(\cdot)$ is defined by:

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} \;:\Longleftrightarrow\; x_1 \cdot x_2 = x_3 \quad \forall \sigma \in S_3, \qquad (2)$$

where $S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $\ell := (13)$, $r := (23)$, $s := (12)$.

Let $\mathfrak{A}$ be a class of quasigroups. A class ${}^{\sigma}\mathfrak{A}$ of σ-parastrophes of all quasigroups from $\mathfrak{A}$ is called σ-parastrophe of $\mathfrak{A}$.

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Functional equations and identities

- Aczél J. "*Lectures on Functional Equations their Applications*" (1966);
- Grätzer "*Universal Algebras*" (1979)

Universal quantified equality is

$$\forall x \forall y \ldots \forall u \quad W = U,$$

where W, U are second order terms and $\{x, y, \ldots, u\}$ is the set of all individual variables appearing in $W = U$.

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Functional equations

A universal quantified equality is said to be a *functional equation*, if it has functional variables.

Pure identities

We will consider only identities which are universal quantified equalities without constants (neither functional nor individual). Every identity defines a variety.

Note, if an identity has a functional or individual constant, then it does not define a variety, because the carrier is defined and the family of functions satisfying the identity forms a set which is empty, if it has no functional variables.

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Pure functional equations

A *pure functional equation* and a *pure identity* are assumed to be the same. Therefore, "*an algebra satisfies a pure identity* $W = U$" and "the algebra is a solution of the pure functional equation" will be the same too.

Quasigroup solutions

The sequence is called *quasigroup solution*, if all functions from the solution are invertible. An equation is called a *quasigroup functional equation*, if it is studied on invertible functions.

A functional equation is said to be *generalized*, if all functional variables are pairwise different.

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Functional equation of distributivity

Generelized distributive functional equation

is a universal quantified equality

$$F_1(F_2(x,y),F_3(x,z)) = F_4(x,F_5(y,z)) \qquad (3).$$

Well-known problem:

find all quasigroup solutions of (3).

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Theorem 1. (Sokhatsky, 2015)

In the class of all loops, an arbitrary identity of parastrophic distributivity defines either trivial variety of loops or the variety of all commutative Moufang loops of degree three.

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Three types of distributivity:

$$x \cdot yz = xy \cdot xz, \qquad \textit{left}$$

$$yz \cdot x = yx \cdot zx, \qquad \textit{right}$$

$$yz \overset{r}{\cdot} x = (y \overset{r}{\cdot} x) \cdot (z \overset{r}{\cdot} x). \qquad \textit{middle}$$

Remind that a *middle translation* of a quasigroup $(Q;\cdot)$ is defined by

$$M_a(x) = y :\Leftrightarrow x \cdot y = a, \qquad M_a(x) = x \overset{r}{\cdot} a.$$

Theorem 2 (Sokhatsky, 2016).

Every two identities of one-side distributivity imply the third one.

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Theorem (V.D. Belousov, 1966)

Every distributive quasigroup is isotopic to a commutative Moufang loop.

Theorem (F.Sokhatsky, 2016)

A quasigroup $(Q;\cdot)$ is distributive if and only if there exists a commutative Moufang loop $(Q;+)$ and its automorphism φ such that $\psi := \iota - \varphi$ is an automorphism of $(Q;+)$, $x\cdot y = \varphi x + \psi y$ and

$$x + (y + z) = (\varphi x + y) + (\psi x + z). \qquad (1)$$

Systems of distributivities

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Moufang loops

$$xy \cdot zx = x(yz \cdot x), \qquad \textit{Moufang identity}.$$

$$L_x(y) \cdot R_x(z) = L_x R_x(yz),$$

$(L_x, R_x, L_x R_x)$ is an autotopism of the loop $(Q; \cdot)$.

Which functional equation of generalized distributivity defines Moufang loops?

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Moufang Loops

Theorem

Let $(Q;\cdot,e)$ be a flexible loop and let

$$f_1(x,y)\cdot zx = f_2(x,yz)$$

hold for some functions f_1 and f_2. Then $(Q;\cdot)$ is a Moufang loop, if $f_1(x,e)=x$.

Example

Let $(Q;\cdot,e)$ be a flexible loop and let

$$y^k(x\cdot y^n)\cdot zx = (x^p\cdot(yz)^p)\cdot(yz\cdot u)$$

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Moufang Loops

Corollary

Let a loop $(Q;\cdot,e)$ satisfy

$$R_x^{-1}(x\cdot yx)\cdot zx = x(yz\cdot x).$$

Then $(Q;\cdot,e)$ is Moufang if and only if it is flexible.

Generalized functional equation of distributivity

$$F_1(F_2(x, y), F_3(x, z)) = F_4(x, F_5(y, z)).$$

If we put $x = a$, then we obtain an isotopy between F_1 and F_5. Renaming the functional variables, we obtain

$$F_1(x, y) \circ F_2(x, z) = F_3(x, y \circ z).$$

Suppose F_1, F_2, F_3 are not necessary to be quasigroups, but right invertible functions. It is easy to see that every triplet of left translations of the operations is an autotopism of the quasigroup $(Q; \circ)$. The reverse is true as well.

Generalized functional equation of distributivity

Theorem

Every solution of the functional equation

$$F_1(x, y) \circ F_2(x, z) = F_3(x, y \circ z)$$

over a quasigroup $(Q; \circ)$ is bijectively defined by an autotopism mapping of the quasigroup.

This theorem reduces the description of all solutions of a functional equation of distributivity to description of the autotopisms of every quasigroup.

Theorem (Belousov, Sokhatsky)

Let $(Q;+,0)$ be an inverse property loop and $M(Q)$ be subloop of Moufang elements. Then a triplet (f_1, f_2, f_3) of operation is a solution of

$$F_1(x,y) + F_2(x,z) = F_3(x, y+z)$$

iff there exist mappings $\alpha, \gamma : Q \to M(Q)$ and binary quasigroup operation h such that
$\gamma(x) + h(x, y+z) = (\gamma(x) + h(x,y)) + h(x,z)$ and

$$f_1(x,y) = \alpha(x) + h(x+y), \quad f_2(x,y) = (h(x,y) + \gamma(x)) + \alpha(x),$$
$$f_3(x,y) = \alpha(x) + (h(x,y) + \gamma(x)) + \alpha(x).$$

Theorem (Sokhatsky)

Let $(Q;+,0)$ be an inverse property loop and the equation

$$F_1(x,y)+F_2(x,z)=F_3(x,y+z)$$

has a solution (f_1,f_2,f_3). If at least one of the operations has at least one permutation among its right translations, then $(Q;+,0)$ is a Moufang loop.

Let inverse property loop $(Q;+,0)$ satisfy an identity

$$(nx+y)+(px+(z+kx))=x+(y+z),\quad n,p,k\in N,$$

then $(Q;+,0)$ is a Moufang loop.

Ternary quasigroups

$$f(x,y,z) = h_1(h_2(x,y),z),$$
$$f(x,y,z) = h_3(h_4(x,y),h_5(z,x)).$$

Which function has both decompositions?

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Which function has both decompositions?

Relation between functional equations

Two equations are called parastrophically-primarily equivalent

if one can be obtained from the other in a finite number of the following steps:

1) application $F({}^{\ell}F(x;y),y)=x$, ${}^{\ell}F(F(x;y),y)=x$, $F(x;{}^{r}F(x;y))=y$, ${}^{r}F(x;F(x;y))=y$;
2) changing sides of the equation;
3) relabeling individual variables;
4) relabeling functional variables;
5) replacing a sub-term $F(x;x)$ with $\delta_F(x)$, if F is diagonal functional variable and vice versa.

A binary functional variable will be called *diagonal*, if it takes its values in the set of all diagonal operations.

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Classification of distributivity-like functional equations

A functional equation is called distributivity-like, if it has three individual variables with 3,2,2 appearances.

Theorem (Sokhatsky, Krainichuk 2009)

Every quasigroup distributivity-like functional equation without squares is parastrophically-primarily equivalent to exactly one of the following equations:

$$F_1(x; F_2(y; z)) = F_3(F_4(x; y); F_5(x; z));$$

$$F_1(y; F_2(x; z)) = F_3(x; F_4(y; F_5(x; z)));$$

$$F_1(y; F_2(x; y)) = F_3(x; F_4(F_5(x; z); z));$$

$$F_1(y; F_2(x; y)) = F_3(F_4(x; z); F_5(x; z));$$

$$F_1(y; F_2(x; z)) = F_3(y; F_4(x; F_5(x; z))).$$

Solutions of distributivity-like functional equations

Let $(Q;\cdot)$ be a group; g be invertible and ${}^{\ell}g\perp(\cdot)$; α, β, γ, δ, μ be permutations of Q; then $(f_1,\ldots,f_5)$ being defined by

$$\begin{gathered} f_1(x;y)=\alpha x\cdot\delta y; \qquad f_2(x;z)=\delta^{-1}\big(g(z;\gamma x)\cdot\gamma x\big); \\ f_3(x;y)=\beta x\cdot\gamma y; \quad f_4(x;y)=\beta^{-1}(\alpha x\cdot\mu y); \\ f_5(x;z)=\mu^{-1}g(z;\gamma x) \end{gathered} \tag{1}$$

is a quasigroup solution of

$$F_1(y;F_2(x;z))=F_3(x;F_4(y;F_5(x;z))). \tag{2}$$

Conversely, if $(f_1,\ldots,f_5)$ is a quasigroup solution of (2), then $\forall e\in Q$ there exists a unique sequence $(\cdot,g,\alpha,\beta,\gamma,\delta,\mu)$ of invertible operations such that $(Q;\cdot,e)$ is a group, $\alpha e=\beta e=\delta e=e$, ${}^{\ell}g\perp(\cdot)$, (1) is valid and

$$\begin{gathered} \alpha x=f_1(x;e), \quad \beta x=f_3(x;{}^{r}f_3(e;e)), \quad \gamma x=f_3(e;x), \\ \delta y=f_1(e;y) \quad \mu x=f_3(f_4(e;x);{}^{r}f_3(e;e)), \\ x\cdot y=f_3(\beta^{-1}x;\gamma^{-1}y), \quad g(z;x)=\mu f_5(\gamma^{-1}x;z). \end{gathered} \tag{3}$$

Corollary 1 for functional equation (2)

Let $(Q; T)$ be an arbitrary topological space and a quintuple $(f_1; \dots; f_5)$ of operations be defined on a set Q by (1), where $(Q; \cdot)$ is a topological group, $(Q; g)$ is a topological quasigroup, ${}^{\ell}g \bot (\cdot)$, α, β, γ, δ, μ are homeomorphisms of $(Q; T)$. Then $(f_1; \dots; f_5)$ is a topological quasigroup solution of the functional equation (2).
Conversely, if a quintuple $(f_1; \dots; f_5)$ of topological quasigroup operations is a solution of (2), then for an arbitrary element $e \in Q$ there exists a single sequence $(\cdot; g; \alpha; \beta; \gamma; \delta; \mu)$ of operations such that $(Q; \cdot)$ is a topological group and e is its neutral element, g is a topological quasigroup operation and ${}^{\ell}g \bot (\cdot)$, α, β, γ, δ, μ are homeomorphisms, $\alpha e = \beta e = \delta e = e$ and (2) are fulfilled. In this case, the sequence $(\cdot; g; \alpha; \beta; \gamma; \delta; \mu)$ is defined by (3).

Corollary 2

Let **R** be the topological space of the real numbers with the natural topology and binary operations $f_1, \dots, f_5$ be defined on **R**. Then a quintuple $(f_1; \dots; f_5)$ is a topological quasigroup solution of the functional equation (2) if and only if there exist homeomorphisms α, β, γ, μ, δ, φ of the space and a topological quasigroup operation g such that ${}^{\ell}g$ is orthogonal to the additive operation $(+)$ of the field **R** and

$$f_1(x; y) = \varphi(\alpha x + \delta y), \quad f_2(x; z) = \delta^{-1}(g(z; \gamma x) + \gamma x),$$
$$f_3(x; y) = \varphi(\beta y + \gamma x), \quad f_4(x; y) = \beta^{-1}(\alpha x + \mu y),$$
$$f_5(x; y) = \mu^{-1} g(y; \gamma x).$$

Solutions of functional equation (3)

Theorem

Let $f_1, \ldots, f_5$ be binary operations, defined on a set Q. Then $(f_1; \ldots; f_5)$ is a quasigroup solution of the functional equation

$$F_1(y; F_2(x; y)) = F_3(x; F_4(F_5(x; z); z))$$

if and only if f_1, f_3 and f_4 are quasigroup operations and there exist permutations α and θ of Q such that the identities

$$f_3(x; \theta x) = \alpha x, \qquad f_2(x; y) = {}^{\ell}f_1(\alpha x; y), \qquad f_5(x; y) = {}^{\ell}f_4(\theta x; y)$$

hold.

Solutions of functional equation (4)

Theorem

Let Q be a set and $f_1, \ldots, f_5$ be binary operations, defined on Q. Then $(f_1; \ldots; f_5)$ is a quasigroup solution of the functional equation

$$F_1(y; F_2(x; y)) = F_3(F_4(x; z); F_5(x; z)),$$

if and only if f_1, f_2 and f_4 are quasigroup operations, $f_1 \perp {}^r f_2$ and there exists a permutation α of Q such that the identities

$$f_3(x; z) = f_1(\alpha^{-1}x; f_2(\alpha^{-1}x; z)); \quad f_5(x; y) = {}^{\ell}f_4(\alpha x; y)$$

hold.

Solutions of functional equation (5)

Theorem

Let Q be a set and $f_1, \dots, f_5$ be binary operations, defined on Q. Then $(f_1; \dots; f_5)$ is a quasigroup solution of the functional equation

$$F_1(y; F_2(x; z)) = F_3(y; F_4(x; F_5(x; z))),$$

if and only if the operations f_2, f_3 and f_5 are quasigroups, $f_2 \perp f_5$ and there exists a permutation α of Q such that the identities

$$f_1(x; y) = f_3(x; \alpha y), \qquad f_4(x; y) = \alpha f_2(x; {}^{r}f_5(x; y))$$

hold.

THANK YOU FOR YOUR ATTENTION!