GALLAI MULTIGRAPHS

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ABSTRACT. A complete edge-colored graph or multigraph is called Gallai if it lacks rainbow triangles. We give a construction of all finite Gallai multigraphs.

1. Background

A complete, edge-colored graph without loops lacking rainbow triangles is called Gallai after Tibor Gallai, who gave an iterative construction of all finite graphs of this sort [3]. Some work progress has been made on the more general problem of understanding edge-colored graphs lacking rainbow \( n \)-cycles for a fixed \( n \). In particular, Ball, Pultr, and Vojtěchovský give algebraic results about the sequence \((n : G \text{ lacks rainbow } n\text{-cycles})\) as a monoid [2], and Vojtěchovský extends the work of Alexeev [1] to find the densest arithmetic progression contained in this sequence [5].

While searching for a general construction of graphs lacking rainbow \( n \)-cycles, we were confronted with the task of understanding a different generalization. We call a complete, simple, edge-colored multigraph Gallai if it lacks rainbow triangles. (By complete here we mean only that each pair of vertices is connected by at least one edge.) Mubayi and Diwan make a conjecture about the possible color densities in Gallai multigraphs having at most three colors [4].

The main result of this paper is a construction of all finite Gallai multigraphs.

1.1. Basic Notation. We denote vertices using lowercase letters such as \( u, v, \) and \( w \), sets of vertices using uppercase letters such as \( U, V, \) and \( W \), and colors using uppercase letters such as \( A, B, \) and \( C \). Given two sets of vertices, \( U \) and \( V \), we write \( UV \) for the set of edges connecting vertices of \( U \) to vertices of \( V \). This notation will also be used with singletons, \( u \) and \( v \), to refer to the edges connecting \( u \) and \( v \). To denote the set of colors present in a set of edges, say \( UV \), we write \( UV \) when there is no risk of ambiguity. Otherwise we refer explicitly to the coloring at hand, i.e. \( \rho(UV) \). If \( UV = \{A\} \), we will often shorten notation by writing \( UV = A \). If the edges of an \( n \)-cycle contain no repeated colors, we say it is rainbow.

Many of our results will be stated in terms of mixed graphs. A mixed graph is a triple \( M = (V, E, A) \) with vertices \( V \), undirected edges \( E \), and directed edges \( A \). We say \( M \) is complete if every pair of distinct vertices is connected by a single directed or undirected edge. The weak components of a directed graph are the components of the graph that results from replacing each directed edge with an undirected edge. For our purposes, the weak components of a mixed graph \( M = (V, E, A) \) will be the weak components of the directed graph \((V, A)\). Note that this notion of component disregards undirected edges.

We use the term rooted tree to refer to a directed graph that is transitive and whose transitive reduction forms a tree in the usual sense. If \((V, A)\) is a rooted tree,
then its root, written $1_V$, is the unique vertex having the property that there is a directed edge from $1_V$ to every other vertex in $V$.

1.2. Construction of Gallai Graphs. It is easy to see that the following construction yields Gallai graphs.

Let $(G = (V, E), \rho)$ be a complete, edge-colored graph such that $|\rho(VV)| \leq 2$. For every $v \in V$, let $(G_v = (V_v, E_v), \rho_v)$ be a Gallai graph. Construct a new complete graph on $\bigcup_{v \in V} V_v$ with edge-coloring $\rho'$ defined by

$$
\rho'(xy) := \begin{cases} \\
\rho_v(xy) & \text{if } x, y \in V_v \\
\rho(uv) & \text{if } x \in V_v, y \in V_w, \text{ and } v \neq w.
\end{cases}
$$

Gallai showed [3] that every finite Gallai graph can be built iteratively by the above construction, i.e. for every Gallai graph, $G = (V, E)$, there exists a nontrivial partition $V = \bigcup V_i$ such that $|V_i V_j| = 1$ for $i \neq j$ and $|\cup_{i \neq j} V_i V_j| \leq 2$.

2. Decomposition of Gallai Multigraphs

While our purpose is to give a construction of Gallai multigraphs, the most substantive step is in developing the appropriate decomposition. Before stating this result, we describe our basic techniques and introduce a few definitions.

2.1. Basic Techniques: Maximality and Dominance. Let $(G = (V, E), \rho)$ be a Gallai multigraph. We will in all cases assume that distinct edges connecting the same vertices are colored distinctly. We also think of $V \subseteq \mathbb{N}$ and thus having a natural ordering. We say that $(G = (V, E), \rho)$ is uniformly colored if $\rho(e_1) = \rho(e_2)$ for all $e_i \in E$.

We call $uv$ isolated if for every $w \notin \{u, v\}$, $uw = uw$ and $|uw| = 1$. Notice that if $uv$ is isolated we can reduce the multigraph by collapsing the edge(s) $uv$. Likewise, given any multigraph, we can arbitrarily introduce new isolated edges without introducing rainbow triangles. We therefore call a multigraph reduced if it contains no isolated edges.

We call $uv$ maximal if no new color can be added to $uw$ without introducing a rainbow triangle. Here we allow the possibility that $uv$ has “all possible colors” and thus is maximal. Likewise, $(G = (V, E), \rho)$ is maximal if $uv$ is maximal for all $u, v \in V$.

Let $(G = (V, E), \rho)$ be a maximal Gallai multigraph. For $u, v \in V$, notice that $|uw| \geq 3$ if and only if $uv$ is isolated. Therefore, if $G$ is reduced, $|uw| = 1$ or $2$ for all $u, v \in V$. Furthermore, if $G$ is not reduced, we can reach a reduced Gallai multigraph by successively collapsing isolated edges of $G$.

To construct all Gallai multigraphs, it therefore suffices to construct the reduced maximal ones. In Section 2.2, we develop a basic decomposition of any maximal reduced Gallai multigraph and then in Section 3 reverse this decomposition to construct all finite reduced Gallai multigraphs.

Lemma 2.1. Suppose $(G = (V, E), \rho)$ is a maximal Gallai multigraph. If $u, v \in V$ and $A \in uw$, then for all $B \notin UV$, there is $w \in V \setminus \{u, v\}$ and $C \notin \{A, B\}$ such that either $A \in uw$ and $C \in uw$ or $C \in uw$ and $A \in uw$.

Proof. Since $G$ is maximal and $B \notin uw$, we can find $w \neq u, v$ such that $u, v, w$ would form a rainbow triangle if $B$ were to be added to $uw$. Thus we may find $X \in uw$ and $Y \in uw$ such that $X, Y$, and $B$ are distinct. However, since $A \in uw$, $|\{X, Y, A\}| \leq 2$ and thus $A = X$ or $A = Y$. Let $C$ be the other color. \(\Box\)
While Theorem 2.2 will follow from Theorem 2.3, our general decomposition result, we present it separately here because of its importance in understanding the most basic structure of a maximal reduced Gallai multigraph.

**Theorem 2.2.** The vertices of a reduced maximal Gallai multigraph that are connected by two edges form uniformly colored cliques.

**Proof.** Let \((G = (V, E), \rho)\) be a reduced maximal Gallai multigraph. Let \(u, v, w \in V\). Suppose \(\overrightarrow{uv} = \{A, B\}\) and \(\overrightarrow{vw} = \{C, D\}\). If \(\{A, B\} \neq \{C, D\}\), then we find a rainbow triangle no matter the colors of \(\overrightarrow{uv}\). Suppose then that \(\overrightarrow{uv} = \overrightarrow{vw} = \{A, B\}\). Certainly \(\overrightarrow{uv} \subseteq \{A, B\}\). Suppose \(\overrightarrow{vw} = A\). Then by Lemma 2.1, we may find \(x \in V \setminus \{u, w\}\) such that, without loss of generality, \(A \in \overrightarrow{uv}\) and \(C \in \overrightarrow{vw}\). Then \(v, x, w\) contains a rainbow triangle. \(\square\)

Theorem 2.3 is primarily an explanation of how each of these uniformly colored cliques are related to each other, and the following relation on sets of vertices plays a central role in this analysis. Let \((G = (V, \mathcal{E}), \rho)\) be a Gallai multigraph. For \(U, V \subseteq V\) disjoint, we say that \(U\) dominates \(V\) and write \(U \rightarrow V\) iff \(|\overrightarrow{UV}| > 0\) and

1. \(U = \{u\}, V = \{v\}\) and \(u < v\) or
2. \(|U| > 1\) or \(|V| > 1\) and for every \(u \in U\) and \(v \in V\), \(\overrightarrow{uv} = \overrightarrow{uv}\).

Given \(U, V \subseteq V\), we write \(\Sigma(U, V)\) for the map from \(U\) to the powerset of \(\overrightarrow{UV}\) defined by \(u \mapsto \overrightarrow{uv}\). When \(U \rightarrow V\), \(\Sigma(U, V)\) completely describes the relationship between \(U\) and \(V\) and we call it the signature of \(U \rightarrow V\).

Given a reduced maximal Gallai multigraph \((G = (V, \mathcal{E}), \rho)\), we will describe its structure through a sequence of edge-colored mixed graphs \(M_n(G) = (V_n, \mathcal{E}_n, \mathcal{A}_n)\) defined as follows:

1. \(V_0 := V\),
2. \(\mathcal{A}_0 := \{((u, v) \in V^2 : u \rightarrow v\},\) and
3. \(\mathcal{E}_0 := \{|u, v| \in |V|^2 : |\rho|uv| = 1\},\)

and for \(n \geq 1\)

1'. \(V_n\) is the partition of \(V\) induced by the weak components of \(M_{n-1}(G)\),
2'. \(\mathcal{A}_n := \{(U, V) \in V_n^2 : U \rightarrow V\},\) and
3'. \(\mathcal{E}_n := \{(U, V) \in [V_n]^2 : |\rho|UV| = 1\}\).

For each \(n\), \(\rho\) induces a list-coloring, \(\rho'\), of \(\mathcal{E}_n \cup \mathcal{A}_n\) by \(\rho'(e) = \rho|UV|\) where \(e = (U, V)\) or \(e = (U, V)\).

For readability, we show only those edges in \(M_n(G)\) that contribute to the formation of directed edges in \(M_{n+1}(G)\). The hash marks on the directed edges in \(M_1(G)\) indicate whether the signatures agree or disagree.

### 2.2. Decomposition of Maximal Gallai Multigraphs

We may now state our main result.

**Theorem 2.3.** Let \(G\) be a maximal Gallai multigraph, \(H\) an induced subgraph of \(G\), and \(M_n(H) = (V_n, \mathcal{E}_n, \mathcal{A}_n)\) the sequence described above. Then

1. \(M_n(H)\) is complete,
2. \(|\rho'(e)| = \begin{cases} 1 & \text{if } e \in \mathcal{E}_n \vspace{0.5em} \text{ or } \mathcal{A}_n \end{cases} \)
3. \(|\rho'(e)| = 2 & \text{if } e \in \mathcal{A}_n \).
Figure 1. Sequence of $M_n(G)$ for a Gallai multigraph.

(3) the weak components of $M_n(H)$ are rooted trees, and
(4) if $(U, V), (V, W) \in \mathcal{A}_n$, then $(U, V) \sim \Sigma (U, W)$
for all $n \geq 0$.

For convenience, if $M_k(H)$ has properties (1)-(4) for all $k \leq n$, we will say that $H$ has the tree property for $n$.

As we are primarily interested in decomposing reduced maximal Gallai multigraphs, our most important application of Theorem 2.3 is when $H = G$. We will, however, need the result in this greater generality in a key technical step in Section 3.2.

2.3. Proof of Theorem 2.3. Throughout this section, we assume $(G = (V, E), \rho)$ is a reduced maximal Gallai multigraph and $H$ is an induced subgraph of $G$.

Lemma 2.4. Suppose $U, V, W \subseteq V$ disjoint, $U \rightarrow V$, and $\{A, B\} = UV$.

1. If $\overline{UV} = C \notin \{A, B\}$, then $\overline{VW} = C$.
2. If $\overline{VW} = C \notin \{A, B\}$, then either $C \in \overline{UW}$ or $U \rightarrow W$ and $\Sigma(U, V) = \Sigma(U, W)$. If we also know that either $U \rightarrow W, W \rightarrow U$, or $|\overline{UW}| = 1$ and that $U$ always dominates with the same colors (i.e., whenever $U \rightarrow U'$, then $\overline{UU'} = \{A, B\}$), then either $\overline{UW} = C$ or $U \rightarrow W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Proof. (1) Fix $v \in V$ and $w \in W$. Since $U \rightarrow V$, we may select $u_A, u_B \in U$ such that $A \in \overline{u_A v}$ and $B \in \overline{u_B v}$. Observe that the triangle $w, u_A, v$ forces $\overline{vwv} \subseteq \{A, C\}$ while $w, u_B, v$ forces $\overline{wv} \subseteq \{B, C\}$. Thus $\overline{vw} = C$. Since $v$ and $w$ were arbitrary, $\overline{VW} = C$. 

Proof. (2)
(2) Fix $u \in U, w \in W, v \in V$. Since $\overrightarrow{UV} = \{A, B\}$. We are in one of the following cases: $\overrightarrow{vw} = A$, $\overrightarrow{uw} = B$, or $\overrightarrow{uw} = \{A, B\}$. If $\overrightarrow{vw} = \{A, B\}$, then $\overrightarrow{uw} = C$ forces $\overrightarrow{vw} = C$. Suppose then that $C \notin \overrightarrow{UV}$. If $\overrightarrow{vw} = A$, then $\overrightarrow{uw} \subseteq \{A, C\}$ and thus $\overrightarrow{vw} = A$. Likewise, if $\overrightarrow{uw} = B$, then $\overrightarrow{uw} = B$. We thus have either $C \in \overrightarrow{UV}$ or $U \rightarrow W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Suppose we also know that we are in one of the following cases:

(i) $U \rightarrow W$ and $\overrightarrow{UV} = \{A, B\}$,
(ii) $W \rightarrow U$, or
(iii) $|\overrightarrow{UV}| = 1$.

Again, if $C \notin \overrightarrow{UV}$, then we must be in case (i). We would like to conclude that if $C \in \overrightarrow{UV}$, then we are in case (iii) and thus $\overrightarrow{UV} = C$. Suppose $W \rightarrow U$. To avoid a rainbow triangle, $\overrightarrow{UV} \subseteq \{A, B, C\}$. In particular, since $|\overrightarrow{UV}| \geq 2$, $A \in \overrightarrow{UV}$ or $B \in \overrightarrow{UV}$. Assume $A \in \overrightarrow{UV}$ and fix $w \in W$ such that $A \in \overrightarrow{wU}$. We may then choose $u \in U$ and $v \in V$ such that $B \in \overrightarrow{uv}$ but now $u, v, w$ is a rainbow triangle. Thus $W \not\rightarrow U$ and $\overrightarrow{UV} = C$.

(2') Let $W = \{w\}$ and $V = \{v\}$ and repeat the proof of (2). \hfill \Box

Lemma 2.5. $H$ has the tree property for 0.

Proof. It is clear that $M_0(H)$ is complete. The rest of the claim is essentially a restatement of Theorem 2.2. By the definition of dominance between single vertices, each complete, uniformly colored clique from Theorem 2.2 becomes a linear ordered set of vertices and thus a rooted tree. In this context, property (4) of Theorem 2.3 is simply the observation that these cliques are uniformly colored. \hfill \Box

Before proceeding, we introduce some convenient notation. Elements of $V_n$ are by definition subsets of $V(H)$. We will however at times want to speak of their structure as rooted trees. For $U \subseteq V_n$, we write $\Upsilon(U)$ to refer to the set of elements of $V_{n-1}$ contained in $U$ and $1_U$ to refer to the root of $\Upsilon(U)$. Notice that $1_U \in V_{n-1}$ has its own tree structure and thus we may refer to $1_U$, $1_{1_U}$, etc. We may continue this recursion until we reach a single vertex. We write $1_U$ to refer to this single vertex. Similarly, for $u \in V(H)$, we write $[u]_n$ to refer to the unique $U \in V_n$ containing $u$. Lastly, we point out how this notation fits together. For $U \subseteq V_n$, $[1_U]^n = U$, $[1_U]^{n-1} = 1_U$, $[1_U]^{n-2} = 1_{1_U}$, ..., and $[1_U]^0 = 1_U$.

We also associate a set of colors with each member of $V_n$ as follows. For $u \in V_0$, $\hat{u} := \cup_{u \rightarrow v} \overrightarrow{vw}$ and for $U \in V_{n+1}$, $\hat{U} := \hat{1}_U$ for $n \geq 0$. Lemma 2.6 demonstrates the importance of this notation.

Lemma 2.6. Suppose $H$ has the tree property for $n$ and $(U, V) \in A_{n+1}$. Then $U$ always dominates with the same two colors $\hat{U}$, i.e.

(1) $\overrightarrow{UV} = \hat{U}$ and
(2) $|\hat{U}| = 2$.

Proof. It is clear that $|\hat{U}| = 2$ since $\hat{U}$ is defined inductively and dominance between two vertices must be with exactly two colors. Likewise, (1) certainly holds for $U, V \in V_0$.

Since $H$ has the tree property for $n$, $|\overrightarrow{UV}| = 1$ for all $U' \in \Upsilon(U)$, and since $U \rightarrow V$, we know $|\overrightarrow{UV}| \geq 2$. Suppose we find $C \in \overrightarrow{UV} \setminus \hat{U}$. Then fix $U_C \in \Upsilon(U)$ such that $\overrightarrow{UCV} = C$. If $U_C = 1_U$, then for all $U' \in \Upsilon(U) \setminus \{1_U\}$, $1_U \rightarrow U'$ and
we may apply part (1) of Lemma 2.4 to get that $\overline{U}\overline{V} = C$ and thus $\overline{U}V = C$, a contradiction.

Suppose then that $1_U \rightarrow U_C$. By induction, $\overline{1_U}U_C = \widehat{U}$. We may now apply part (2) of Lemma 2.4 to get that either $C \in \overline{1_U}1_V$ or $1_U \rightarrow 1_V$. The former has already been ruled out while the latter contradicts the assumption that $1_U$ and $1_V$ were in different components of $\mathcal{V}_n$.

The following lemma is useful because it allows us to locate a vertex in $U$ that is connected to the rest of $U$ by only the colors contained in $\widehat{U}$.

Lemma 2.7. If $H$ has the tree property for $n$ and $U \in \mathcal{V}_{n+1}$, then $\overline{U}1_U = \widehat{U}$.

Proof. By Lemma 2.6, $|\widehat{U}| = 0$ or 2. If $\widehat{U} = \emptyset$, then $U$ is a single vertex, i.e. $U = \{1_U\}$, and thus $\overline{U}1_U = \emptyset$.

For $n = 0$, $U$ is either a single vertex, in which case $\widehat{U} = \emptyset$, or $U$ is a nontrivial uniformly colored clique, in which case $\widehat{U}$ is by definition $\overline{1_U}U$.

For $n \geq 1$, if $U$ is a single vertex, again $\widehat{U} = \emptyset$. Otherwise, by induction $\overline{1_U}1_U = 1_U = \widehat{U}$. But since $1_U = 1_U$, we have $\overline{1_U}1_U = \widehat{U}$, and by Lemma 2.6, $\overline{1_U}1_{U'} = \widehat{U}$ for every $U' \in \mathcal{Y}(U) \setminus \{1_U\}$. Finally, given that $1_U \in 1_U$, $\overline{1_U}1_{U'} \subseteq \overline{1_U}1_{U'}$ and thus $\overline{1_U}1_{U'} = \widehat{U}$.

Lemmas 2.8 and 2.9 will be used in situations where a tree is connected to another tree or vertex by a color not present in the dominating colors of the first tree.

Lemma 2.8. Suppose $H$ has the tree property for $n$, $U, V \in \mathcal{V}_{n+1}$ distinct, and $V' \in \mathcal{Y}(V)$ such that $C \in \overline{U}V' \setminus \widehat{U}$. Then $\overline{U}V' = C$.

Proof. If $\overline{1_U}V' = C$, then for every $U' \in \mathcal{Y}(U) \setminus \{1_U\}$ we may apply part (1) of Lemma 2.4 with $1_U \rightarrow U'$ and $V'$ to get that $\overline{U}V' = C$ and thus $\overline{U}V' = C$. Suppose then that $U' \in \mathcal{Y}(U) \setminus \{1_U\}$ such that $\overline{U}V' = C$. Applying part (2) of Lemma 2.4 with $1_U \rightarrow U'$ and $V'$, we get that $\overline{1_U}V' = C$ or $1_U \rightarrow V'$. Since $1_U$ and $V'$ are in different components of $\mathcal{V}_{n+1}$, we must be in the former case, and by the previous case we are done.

Lemma 2.9. Suppose $H$ has the tree property for $n$, $U \in \mathcal{V}_{n+1}$, and $v \in V$ such that $\overline{1_U}V = C \not\in \widehat{U}$. Then $\overline{1_U}V = C$.

Proof. First observe that $v \not\in U$ since by Lemma 2.7, $\overline{1_U}U = \widehat{U}$. Next let $k$ be maximal such that $[1_U]_{k}v = C$. If $k = n + 1$, we are done. Suppose $k < n + 1$ and select $u \in [1_U]_{k+1} \setminus [1_U]_k$. We may apply (1) of Lemma 2.4 with $[1_U]_{k}u$ and $v$ to get that $\overline{v}v = C$ and thus violating the maximality of $k$.

Note that in Lemma 2.9 we do not require that $v$ be in $V(H)$ but rather in the larger set $V$.

Lemma 2.10. For $n \geq 0$, suppose $H$ has the tree property for $n$ and $U, V \in \mathcal{V}_{n+1}$ such that $\overline{U}V \subseteq \widehat{U} = \overline{V} = \{A, B\}$ Then the following statements are equivalent:

1. $U \rightarrow V$,
2. there is $x \in V \setminus (U \cup V)$ such that $U \rightarrow x$ and $\overline{V}x = C \not\in \{A, B\}$, and
3. $V \not\rightarrow U$ and $\overline{U}V = \{A, B\}$.

If the statements are true, then $\Sigma(U, V) = \Sigma(U, x)$. 
Proof. (1 $\Rightarrow$ 2) We may assume $\overline{1_U V} = A$. Since $G$ is maximal and $B \notin \overline{1_U V}$, there is $x \in V$ such that $A \subseteq \overline{1_U x}$ and $C \subseteq \overline{x V}$ or $C \subseteq \overline{1_U x}$ and $A \subseteq \overline{x V}$ for $C \notin \{A, B\}$. Notice that if $C \subseteq \overline{1_U x}$, since $C \notin \overline{U x}$, $\overline{1_U x}$ cannot contain multiple edges and thus $C = \overline{1_U x}$. Likewise, if $C \subseteq \overline{1_V x}$. Furthermore, in either case, since $C \notin \overline{1_U U} = \overline{1_V V}$, we must conclude that $x \in V \setminus (U \cup V)$. Suppose we are in the latter case, i.e. $C = \overline{1_U x}$ and $A \subseteq \overline{x V}$. By Lemma 2.9, $\overline{U x} = C$. We may then apply part (1) of Lemma 2.4 with $U \rightarrow V$ and $x$ to get that $\overline{V x} = C$, which contradicts our assumption that $A \in \overline{1_V x}$.

We must then be in the former case, i.e. $A \subseteq \overline{1_U x}$ and $C \subseteq \overline{x V}$ and, again by Lemma 2.9, $\overline{V x} = C$. Note that if we can show that $C \notin \overline{U x}$, we may then apply part (2) of Lemma 2.4 with $U \rightarrow V$ and $x$ to get that $U \rightarrow x$ and that $\Sigma(U, V) = \Sigma(U, x)$, which is exactly what we would like to prove.

To this end, suppose $C \subseteq \overline{U x}$ and let $k$ be minimal such that $C \subseteq \overline{1_U k x}$. If $k = 0$, we have that $C \subseteq \overline{1_U x}$ and it again follows that $\overline{U x} = C$, which is a contradiction. Thus $k > 0$. Fix $u \in \overline{1_U x}$ such that $C \subseteq \overline{x u}$. Since $k$ is minimal, $u \in \overline{1_U k \setminus 1_U k-1}$ and thus $1_U k-1 \rightarrow u$. Recall that $[1_U]_i = \hat{U}$ for all $i \leq n$ and thus $[1_U]_{k-1} u = \{A, B\}$.

We may now apply part (2') of Lemma 2.4 with $1_U k-1 \rightarrow u$ and $x$ to get that either $C \subseteq \overline{1_U k-1 x}$ or $1_U k-1 \rightarrow x$ and $\Sigma([1_U]_{k-1}, 1_U) = \Sigma([1_U]_{k-1}, u)$. By the minimality of $k$, we must be in the latter case. Then we may apply part (2) of Lemma 2.4 with $1_U k-1 \rightarrow x$ and $V$ to get that either $C \subseteq \overline{1_U k-1 V}$ or $1_U k-1 \rightarrow V$. Both of these cases contradict the assumption that $1_U V = A$. Thus $C \notin \overline{U x}$.

(2 $\Rightarrow$ 3) We may apply part (2) of Lemma 2.4 with $U \rightarrow x$ and $V$ to get that either $C \subseteq \overline{U V}$ or $U \rightarrow V$ and $\Sigma(U, x) = \Sigma(U, V)$. Since $C \notin \overline{U V} \subseteq \{A, B\}$, we are left with the latter case; $U \rightarrow V$ and thus $\overline{U V} = \hat{U} = \{A, B\}$ and $V \notin U$.

(3 $\Rightarrow$ 1). Again we may assume $\overline{1_V V} = A$. As argued in (1 $\Rightarrow$ 2), we may find $x \in V \setminus (U \cup V)$ such that either $A \subseteq \overline{1_V x}$ and $\overline{V x} = C$ or $U x = C$ and $A \subseteq \overline{1_V x}$ for some $C \notin \{A, B\}$. Suppose we are in the latter case. Since $V \notin U$ and $\overline{U V} = \{A, B\}$, there must be $U_A, U_B \in \Upsilon(U)$ and $V' \in \Upsilon(V)$ such that $\overline{U_A V'} = A$ and $\overline{U_B V'} = B$. Then $x, U_A, V'$ forces $\overline{x V'} \subseteq \{A, C\}$ while $x U_B V'$ forces $\overline{x V'} \subseteq \{B, C\}$. Thus $\overline{x V'} = C$ and $C \subseteq \overline{V x}$.

We may then let $k$ be minimal such that $C \subseteq [1_V]_{k x}$. If $k = 0$, then $C \subseteq \overline{1_V x}$ and $\overline{V x} = C$, which contradicts our assumption that $A \subseteq \overline{1_V x}$. Therefore $k > 0$. As before, we select $v \in [1_V]_k \setminus [1_V]_{k-1}$ such that $C \subseteq \overline{x v}$. We now apply part (2') of Lemma 2.4 with $[1_V]_{k-1} \rightarrow v$ and $x$ to get that either $C \subseteq \overline{[1_V]_{k-1} x}$ or $[1_V]_{k-1} \rightarrow x$. By the minimality of $k$, we must be in the latter case and we may apply part (2) of Lemma 2.4 with $[1_V]_{k-1} \rightarrow x$ and $1_U$ (recall our assumption that $\overline{U x} = C$) to get that either $C \subseteq \overline{1_U [1_V]_{k-1}}$ or $[1_V]_{k-1} \rightarrow 1_U$. Both of these cases contradict the assumption that $1_U V = A$.

We therefore may assume that $A \subseteq \overline{1_U x}$ and $\overline{V x} = C$. It is either the case that $U \rightarrow V$ or $U \notin V$. If we suppose that $U \notin V$, then we are in the case just handled with the roles of $U$ and $V$ reversed. Since that assumption leads to a contradiction, we have that $U \rightarrow V$.

Lemma 2.11. If $H$ has the tree property for $n$, then $M_{n+1}(H)$ is complete.
Proof. Let \( U, V \in \mathcal{V}_{n+1} \). If \( |UV| = 1 \), then \( \{U, V\} \in \mathcal{E}_{n+1} \). Suppose then that \( |UV| > 1 \). Notice that if \( \tilde{U} = \emptyset \), then \( U \) is a single vertex and thus \( V \to U \) and \( (V, U) \in \mathcal{A}_{n+1} \).

We may therefore assume \(|\tilde{U}| = |\tilde{V}| = 2\) and consider the following cases:

Case 1: \( \tilde{U} \neq \tilde{V} \) and \( |UV| > 2 \). We may then select \( C_U, C_V \in UV \) distinct such that \( C_U \notin \tilde{U} \) and \( C_V \notin \tilde{V} \) and \( U' \in \mathcal{Y}(U), V' \in \mathcal{Y}(V) \) such that \( C_V \notin UV \) and \( C_U \in UV' \). By Lemma 2.8, \( UV = C_V \) and \( UV' = C_U \). This implies that \( C_V = UV = C_U \), a contradiction.

Case 2: \( \tilde{U} \neq \tilde{V} \) and \( |UV| = 2 \). Then we may assume there is \( C \in UV \) such that \( C \notin \tilde{U} \). Let \( UV = \{C, D\} \). Select \( V_C \in \mathcal{Y}(V) \) such that \( C \in UV_C \). By Lemma 2.8, \( UV_C = C \). Now select \( V_D \in \mathcal{Y}(V) \) such that \( D \in UV_D \). Observe that \( C \notin UV_D \) since that would imply that \( D \notin UV_D = C \). Thus \( UV_D = D \). Since \( UV = \{C, D\} \), we have accounted for every element of \( \mathcal{Y}(V) \) and \( V \to U \), i.e. \( (V, U) \in \mathcal{A}_{n+1} \).

Case 3: \( \hat{U} = \hat{V} = \{A, B\} \) and \( C \in UV \). Fix \( U_C \in \mathcal{Y}(U) \) such that \( C \in UC \). By Lemma 2.8, \( UC = C \) and thus for every \( V' \in \mathcal{Y}(V), C \in UV' \). Applying Lemma 2.8 again, gives us that \( UV = C \) and thus \( \mathcal{U} = C \). Thus \( \{U, V\} \in \mathcal{E}_{n+1} \).

Case 4: \( UV = \hat{U} = \hat{V} \). If \( V \not\to U \), apply (3 \( \Rightarrow \) 1) from Lemma 2.10 to get that \( U \to V \). \( \square \)

Lemma 2.12. Suppose \( H \) has the tree property for \( n \). The weak components of \( M_{n+1}(H) \) are transitive, and if \( (U, V), (V, W) \in \mathcal{A}_{n+1} \), then \( (U, V) \sim_{\Sigma} (U, W) \).

Proof. Let \( U, V, W \in \mathcal{V}_{n+1} \) such that \( U \to V \) and \( V \to W \). By Lemma 2.6, \( |\hat{U}| = |\hat{V}| = 2 \). We consider two cases: \( \hat{U} \neq \hat{V} \) and \( \hat{U} = \hat{V} \).

Suppose \( \hat{U} \neq \hat{V} \) and let \( A \in \hat{U} \setminus \hat{V} \). Fix \( U_A \in \mathcal{Y}(U) \) such that \( UA = A \) and \( V_1, V_2 \in \mathcal{Y}(V) \) such that \( V_1W \neq V_2W \). Fix \( W' \in \mathcal{Y}(W) \). We have that \( U_A, V_1, W' \) forces \( UA W' \subseteq \{A, V_1W\} \) while \( U_A, V_2, W' \) forces \( UA W' \subseteq \{A, V_2W\} \) and thus \( UA W' = A \). Since \( W' \) was arbitrary, \( UA W = A \). Note that since \( |UA W| = 1 \), we have ruled out the possibility that \( W \to U \). By Lemma 2.11, we will be done if we can show that \( |UV| > 1 \). Observe that we could also choose \( U_B \in \mathcal{Y}(U) \) such that \( UB W = B \neq A \). If it happens that \( B \notin \hat{V} \), by the same reasoning as above \( UB W = B \) so that \( \{A, B\} \subseteq UU \) and thus \( U \to W \) and \( \Sigma(U, V) = \Sigma(U, W) \).

Suppose then that \( \hat{U} = \{A, B\} \) and \( \hat{V} = \{B, C\} \). We can now find \( U_A, U_B \in \mathcal{Y}(U) \) such that \( U_A W = A \) and \( U_B W \subseteq \{B, C\} \). Therefore \( |UV| > 1 \) and by Lemma 2.11 we have that \( U \to W \). By Lemma 2.6, \( UW = \hat{U} = \{A, B\} \) and thus \( UB W = B \). Therefore, \( \Sigma(U, V) = \Sigma(U, W) \).

We now consider the case \( \hat{U} = \hat{V} = \{A, B\} \). Note that \( UV \subseteq \{A, B\} \) since we may otherwise easily form a rainbow triangle. We now have the setup for (1 \( \Rightarrow \) 2) of Lemma 2.10 with \( U \to V \) and have \( x \in V \setminus (U \cup V) \) such that \( U \to x \), \( \Sigma(U, V) = \Sigma(U, x) \), and \( xW = C \neq \{A, B\} \). Applying part (1) of Lemma 2.4 to \( V \to W \) and \( x \), we have that \( xW = C \). Now apply part (2) of Lemma 2.4 with \( U \to x \) and \( W \) to get that either \( C \in UV \) or \( U \to W \) and \( \Sigma(U, W) = \Sigma(U, x) = \Sigma(U, V) \). We have already ruled out the former while the latter is what we sought to prove. \( \square \)

Lemma 2.13. Suppose \( H \) has the tree property for \( n \). The weak components of \( M_{n+1}(H) \) form rooted trees.
Proof. After Lemma 2.12, we need only show that for $U_1, U_2, V \in \mathcal{V}_{n+1}$ distinct, if $U_1 \rightarrow V$ and $U_2 \rightarrow V$, then either $U_1 \rightarrow U_2$ or $U_2 \rightarrow U_1$. By Lemma 2.11, it suffices to show $|U_1 U_2| > 1$.

Suppose $|U_1 U_2| = 1$. Observe that if $|U_1 V \cup U_2 V \cup U_1 U_2| > 2$, then we must find a rainbow triangle in $U_1, U_2, V$. Thus we may assume $U_1 U_2 = A \in \hat{U}_1 = \hat{U}_2 = \{A, B\}$. As in the proof of Lemma 2.10, since $\hat{U}_1 = \hat{U}_2$, we may select $x \in V \setminus (U_1 \cup U_2)$ such that $\hat{U}_1 x = C \neq \{A, B\}$ and $A \notin \hat{U}_2 x$ (or with the roles of $U_1$ and $U_2$ reversed).

We may then apply part (1) of Lemma 2.4 with $U_1 \rightarrow V$ and $x$ to get that $V x = C$ and apply part (2) of Lemma 2.4 with $U_2 \rightarrow V$ and $x$ to get that either $C \in \hat{U}_2 x$ or $U_2 \rightarrow x$.

First we consider the case $C \in \hat{U}_2 x$. Let $k$ be minimal such that $C \in [1_{U_2}] k x$. If $k = 0$, by Lemma 2.9, $\hat{U}_2 x = C$, which contradicts our assumption that $A \notin \hat{U}_2 x$. Thus $k > 0$ and we may select $u \in [1_{U_2}] k \setminus [1_{U_2}] k-1$ such that $C \in C\{u\}$. We may apply part (2') of Lemma 2.4 with $[1_{U_2}] k-1 \rightarrow u$ and $x$ to get that either $C \in [1_{U_2}] k x$ or $[1_{U_2}] k-1 \rightarrow x$. By the minimality of $k$, we must be in the latter case. However, by assumptions that $\hat{U}_1 U_2 = A$ and $\hat{U}_1 x = C$ and thus we may locate a rainbow triangle in $U_1, [1_{U_2}] k-1, x$.

We turn now to the second case, $U_2 \rightarrow x$. We may apply part (2) of Lemma 2.4 with $U_2 \rightarrow x$ and $U_1$ to get that either $C \in \hat{U}_1 U_2$ or $U_2 \rightarrow U_1$, which both contradict our assumption that $U_1 U_2 = A$. Thus $U_1 \rightarrow U_2$ or $U_2 \rightarrow U_1$.

Taking Lemmas 2.11, 2.12, and 2.13 together we have proved Theorem 2.3.

3. Construction of Finite Gallai Multigraphs

In Section 2.2, we found that any reduced maximal Gallai multigraph $(G, \rho)$ can be decomposed into a sequence of mixed graphs, $(M_k(G), \rho', \sim \Sigma)$, having certain properties. We now reverse this process to construct all finite Gallai multigraphs. An example of this construction is presented in Figure 2.

Construction 1 (Edge-colored Multigraph Construction $\Gamma$). Given a triple $(M = (V, E, A), \rho', \sim \Sigma)$ with $M$ a complete mixed graph, $\rho'$ a list coloring of $E \cup A$ such that $|\rho'(e)| = 1$ for $e \in E$ and $|\rho'(e)| = 2$ for $e \in A$, and $\sim \Sigma$ an equivalence on members of $A$ sharing an initial vertex, construct a complete, edge-colored multigraph $(G = (V, E), \rho)$ as follows:

(1) Replace each $u \in V$ with $(G_u = (E_u, V_u), \rho_u)$, a uniformly colored Gallai multigraph such that if $(u, v) \in E$ for some $v \in V$, then $|V_u| \geq 2$ and $\rho_u[V_u V_u] = \rho'((u, v))$.

(2) For each $u \in V$, $\rho |_{E_u} = \rho_u$.

(3) For $u, v \in V$ distinct, connect $V_u$ and $V_v$ as follows:

(a) if $(u, v) \in E$, then $\rho((w_1, w_2)) = \rho'((u, v))$ for $w_1 \in V_u$ and $w_2 \in V_v$;

(b) if $(u, v) \in A$, then define $\rho$ such that

(i) $V_u \rightarrow V_v$;

(ii) $\rho[V_u V_v] = \rho'((u, v))$;

(iii) $\Sigma(V_u, V_v) = \Sigma(V_v, V_u)$ whenever $(u, v) \sim \Sigma (u, w)$.

It is easy to see that $\rho$ can be defined in this way whenever $|V_u| \geq 2$ as required above.
We write $\Gamma\left((M, \rho', \sim_{\Sigma})\right)$ for the family of edge-colored multigraphs resulting from all possible choices of $\{(G_u, \rho_u)\}$ in step (1) and permissible definitions of $\rho$ in step (3.b).

We use the following notation in the construction of Gallai multigraphs.

- $G = \{(G_i, \rho_i)\}$ is the family of all finite Gallai multigraphs.
- $G_r = \{(G_i, \rho_i)\}$ is the family of all finite reduced Gallai multigraphs.
- $G_r^+$ is the family of all induced subgraphs of maximal members of $G_r$.
- $M = \{(M_n(H), \rho, \sim_{\Sigma}) : H \in G_r^+ \text{ and } n \in \mathbb{N}\}$.
- $T$ is the family of rooted trees in $M$.
- $G_r^*$ is the family of all induced subgraphs of maximal members of $G_r$.
- $T^*$ is the family of rooted trees in $G_r^*$.

Lastly, we let $G(n) = \{(G = (V, E), \rho) : |V| \leq n\}$ and likewise for each of the families defined above. We will construct a family $M'$ and show that

- $M \subseteq M' \subseteq M^*$ and
- $G_r^+ \subseteq \Gamma[M] \subseteq \Gamma[M'] \subseteq \Gamma[M^*] \subseteq G$.

In particular, we will construct a family of Gallai multigraphs $\Gamma[M']$, containing the reduced maximal ones.

**Lemma 3.1.** Suppose $(M_n(G) = (V, A, E), \rho, \sim_{\Sigma}) \in M$. If $\rho'$ is a coloring of $A \cup E$ such that $\rho'(e) \in \rho(e)$ for all $e \in A \cup E$ and $\rho'(e_1) = \rho'(e_2)$ whenever $e_1 \sim_{\Sigma} e_2$, then $(M_n(G), \rho')$ lacks rainbow triangles.

**Proof.** Fix $U, V, W \in V$ distinct. Recall that $U, V,$ and $W$ are disjoint subsets of vertices of $G$ and that $(U, V) \in A$ if and only if $U \to V$ in $G$. We now consider each of the general cases presented in Figure 3. Fix $v \in V$ and $w \in W$.

Case (A): Select $u \in U$ and note that $\rho'$ colors the triangle $U, V, W$ just as $u, v, w$ is colored in $G$.

Case (B): Select $u \in U$ such that $uV = \rho'(U, V)$. Again, $U, V, W$ is colored in the same way as $u, v, w$.

Cases (C) and (E): Since $(U, V) \sim_{\Sigma} (U, W)$, $\rho'((U, V)) = \rho'((U, W))$.

Case (D): Since $(U, V) \not\sim_{\Sigma} (U, W)$, we know $\Sigma(U, V) \neq \Sigma(U, W)$ and we may thus select $u \in U$ such that $uV \neq uW$. It might happen that $uV = \rho'((U, V))$ and $uW = \rho'((U, W))$. In this case, we again note that $\rho'$ colors $U, V, W$ in the same way that $u, v, w$ is colored in $G$. Suppose then that we are in the alternate
case: \( \overline{uV} = \rho'(\{U, W\}) \) and \( \overline{uW} = \rho'(\{U, V\}) \). This however also forces \( \overline{VW} = \rho'(\{V, W\}) \} \in \{\rho'(\{U, V\}), \rho'(\{U, W\})\}\}. \hfill \Box

**Lemma 3.2.** \( G^+ \subseteq \Gamma[\mathcal{M}] \subseteq \mathcal{G} \) and thus \( \mathcal{M} \subseteq \mathcal{M}^* \).

*Proof.* To see that \( \Gamma[\mathcal{M}] \subseteq \mathcal{G} \), let \( \mathcal{G} = (V, A, \mathcal{E}, \mathcal{S}, \rho, \sim_{\mathcal{S}}) \in \mathcal{M} \) and fix \( G = (V, E, \rho') \in \Gamma((M, \rho, \sim_{\mathcal{S}})) \). Let \( \{G_u = (E_u, V_u), \rho_u\} \) be the family of Gallai multigraphs used in step (1) of the construction of \( (G, \rho') \). We now show that \( (G, \rho') \) lacks rainbow triangles.

First note that for a given triangle, \( u, v, w \), if any two of the vertices correspond to the same vertex of \( M \), i.e. fall in the same \( V_i \), then \( u, v, w \) must have a repeated color.

Therefore we only need to consider the triangles formed by vertices from different \( G_i \). In particular, we may form \( V' \subseteq V \) by selecting a single vertex from each of \( V_i \) and consider \( (V', E', \rho') \), the induced graph of \( G \) by \( V' \).

We will be done if we show that \( (V', E', \rho') \) lacks rainbow triangles. Notice that this triple is equivalent to an edge-coloring of \( M(G) \) of the form described in Lemma 3.1 and thus \( (V', E', \rho') \) lacks rainbow triangles.

Finally, note that \( H \in \Gamma((M_1(G), \rho_1, \sim_{\mathcal{S}})) \) for each \( H \in G^+ \) and thus \( G^+ \subseteq \Gamma[\mathcal{M}] \). \hfill \Box

We now construct a family \( \mathcal{M}' \) such that \( \mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}^* \).

**Construction 2 (Forest Construction \( \Delta_{\mathcal{F}} \)).** Suppose we have at our disposal \( \mathcal{N} \subseteq \mathcal{M}^* \). Set \( \mathcal{T}_\mathcal{N} = \mathcal{N} \cap \mathcal{T} \) and form a triple \( (\mathcal{M}' = (V', E', A'), \rho', \sim_{\mathcal{S}'}) \) as follows:

1. Fix \( (M = (V, E, A), \rho, \sim_{\mathcal{S}}) \in \mathcal{N} \).
2. For each \( u \in V \), fix \( (T_u = (V_u, E_u, A_u), \rho_u, \sim_{\mathcal{S}}) \in \mathcal{T}_\mathcal{N} \) such that if \( (u, v) \in A \) for some \( v \in V \), then
   (a) \( T_u = \rho((u, v)) \) (this is just an issue of labeling) and
   (b) \( |V_u| \geq 2 \).
3. Set \( V' := \bigcup_{u \in V} V_u \).
4. Set \( A' := \bigcup_{u \in V} A_u \).
5. Set \( E' := \bigcup_{u \in V} E_u \bigcup \{\{w_1, w_2\} : w_1 \in V_u, w_2 \in V_v, \text{ and } u \neq v\} \).
6. Define \( \rho' \) as follows:
   (a) \( \rho'(A_u \cup E_u) := \rho_u \) for all \( u \in V \);
   (b) for \( \{w_1, w_2\} \in E' \) where \( w_1 \in V_u, w_2 \in V_v, \text{ and } u \neq v \),
      (i) if \( (u, v) \in E \), then \( \rho'(\{w_1, w_2\}) := \rho(\{u, v\}) \);
      (ii) if \( (u, v) \in A \), then define \( \rho' \) on \( V_u V_v \) so that
          (A) \( V_u \to V_v \).
(B) $\rho'[V_uV_v] = \rho((u,v))$, and
(C) $\Sigma(V_u, V_v) = \Sigma(V_w, V_w)$ whenever $(u,v) \sim \Sigma (u,w)$

Key to this construction is understanding when and how step (6.b.ii) can be accomplished. We call this the signature configuration problem and resolve it in Section 3.1

We write $\Delta_F(\mathcal{N})$ for the family of all mixed graphs resulting from one iteration of this construction beginning with $\mathcal{N}$. Notice that $\Delta_F(\mathcal{N})$ constructs no rooted trees that were not already present in $\mathcal{N}$, and we will therefore need a separate construction for trees. We present this tree construction in Section 3.2 and write $\Delta_T(\mathcal{N})$ for the resulting family of trees. Finally, we set $\Delta(\mathcal{N}) := \Delta_F(\mathcal{N}) \cup \Delta_T(\mathcal{N})$.

We let $\mathcal{M}_0$ be the family of elements in $\mathcal{M}$ having no directed edges (and are thus essentially Gallai graphs) and for $n \geq 0$, set $\mathcal{M}_{n+1} := \Delta(\mathcal{M}_n)$ and $\mathcal{M}' := \cup_{n=0}^\infty \mathcal{M}_n$.

**Theorem 3.3.** $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}^*$ and thus $\mathcal{G}_r^+ \subseteq \Gamma(\mathcal{M}') \subseteq \mathcal{G}$.

We prove Theorem 3.3 after addressing the signature configuration problem in Section 3.1 and describing the tree construction $\Delta_T$ in Section 3.2.

3.1. **Signature Configuration Problem.** Here we address the following basic problem arising in step (6.b.ii) of the $\Delta_F$ construction: given $T \in \mathcal{T}$ and a set of vertices $U$, describe all possible ways to join $T$ and $U$ with two colors so that $T \rightarrow U$ and $T \cup U$ lacks rainbow triangles.

First observe that, since $T \rightarrow U$, the set of vertices $U$ is irrelevant and that we may just as well assume there is only a single vertex $u$. Now suppose we would like to define dominance between $T = (V,E,A)$ and $u$ using the colors $A$ and $B$. Up to relabeling, we may assume $\hat{T} = \{A,B\}$. The signature of $T \rightarrow u$ is determined by the map $\Sigma : V \rightarrow \{A,B\}$ given by $v \mapsto \overline{v_u}$.

For $v_1, v_2 \in V$, if $\overline{v_1} \overline{v_2} \not\subseteq \{A,B\}$, then to avoid a rainbow triangle it must be the case that $\Sigma(v_1) = \Sigma(v_2)$, i.e. $\overline{v_1 u} = \overline{v_2 u}$. We therefore partition $V$ by first defining the relation $v_1 \sim v_2$ if $\overline{v_1} \overline{v_2} \not\subseteq \{A,B\}$ and then extending $\sim$ to an equivalence.

**Lemma 3.4.** $T \cup u$ lacks rainbow triangles if and only if $\Sigma$ is constant on $\sim$ classes.

**Proof.** The statement follows directly from the definition of $\sim$. □

**Theorem 3.5.** For $T = (V,E,A) \in \mathcal{T}$ and $M \in \mathcal{M}$, the ways of joining $T$ and $M$ so that $T \rightarrow M$ and $T \cup M$ lacks rainbow triangles correspond exactly to choice functions $\Sigma : (V/\sim) \rightarrow \hat{T}$.

**Proof.** The only difference between this statement and Lemma 3.4 is that $T$ is now dominating a set of vertices rather than a single vertex, but since $T \rightarrow M$, whatever colors may be present in $M$ cannot form rainbow triangles with vertices from $T$. The requirement that $\Sigma$ be onto corresponds to the fact that dominance requires multiple colors and $|\hat{T}| = 2$. □

3.2. **Tree Construction $\Delta_T$.** We now describe how to construct rooted trees of size $n + 1$ out of a mixed graph of size $n$. Here we use the fact that Theorem 2.3 holds for all of $\mathcal{G}_r^+$, rather than just the maximal members.

We would like to construct $T = (V,E,A) \in \mathcal{T}$ using elements of $\mathcal{M}$ having fewer than $|V|$ vertices. Perhaps the most natural approach would be to consider the collection of subtrees $\{T_i\}$ in $V \setminus 1_T$ as in the left-most image of Figure 4 and then
list all possible ways to assign signatures to $1_T \rightarrow T_i$. One quickly realizes that with this approach you must also describe how each pair of $\{T_i\}$ is related.

Notice, however, that since $T \in T \subseteq \mathcal{M}$, $V$ corresponds to the induced subgraph of some reduced maximal Gallai multigraph, and therefore $V \setminus 1_V$ does as well. We may thus use Theorem 2.3 to show that $V \setminus 1_V$ decomposes into a mixed graph in $\mathcal{M}$. In particular, Lemma 3.6 shows that if we remove the root from $T$, we are left with $M \in \mathcal{M}$ whose weak components are $\{T_i\}$, as in the center image of Figure 4. We may now collapse $M$ into $M'$, as in the right-most image of Figure 4.

**Lemma 3.6.** Let $(G = (V, E, \rho)) \in \mathcal{G}_r^+$ and $M_k(G) = (\mathcal{V}_k, \mathcal{E}_k, \mathcal{A}_k)$ for all $k \leq n$. Let $V_T \in \mathcal{V}_n$ and let $H$ be the induced subgraph of $G$ by $V \setminus V_T$. Let $M_k(H) = (\mathcal{V}_k', \mathcal{E}_k', \mathcal{A}_k')$ for all $k \leq n$. Set $V_T(k) = \{W \in \mathcal{V}_k : W \subseteq V_T\}$. Then $\mathcal{V}_k = \mathcal{V}_k' \cup V_T(k)$ for all $k \leq n$.

**Proof.** The claim holds for $k = 0$ since $\mathcal{V}_0 = V$, $\mathcal{V}_0' = V \setminus V_T$, and $V_T(0) = V_T$. Let $k + 1 \leq n$ be minimal such that $\mathcal{V}_{k+1} \neq \mathcal{V}_{k+1}' \cup V_T(k + 1)$. By definition, $V_T(k + 1)$ agrees with $\mathcal{V}_{k+1}$ on the $V_T$ portion of $V$. It must be the case then that $\mathcal{V}_{k+1} \setminus V_T(k + 1) \neq \mathcal{V}_{k+1}' \cup V_T(k + 1)$. That is, $\mathcal{V}_{k+1} \setminus V_T(k + 1) \neq \mathcal{V}_k \setminus V_T(k) = \mathcal{V}_k'$. Note that $\mathcal{V}_{k+1}'$ is completely determined by the dominance relations between members of $\mathcal{V}_k'$. Likewise, since $V_T \in \mathcal{V}_n$, none of the members of $\mathcal{V}_k'$ is in a dominance relation with any member of $V_T(k)$ and thus $\mathcal{V}_{k+1} \setminus V_T(k + 1)$ is also determined by the dominance relations between members of $\mathcal{V}_k'$. Thus $\mathcal{V}_{k+1} \setminus V_T(k + 1) = \mathcal{V}_{k+1}'$, a contradiction.

We will be done once we specify all ways in which we can assign signatures to the directed edges from $1_T$ to the vertices of $M'$. In particular, a choice of signatures corresponds exactly to a partition of the vertices of $M'$. What then are the necessary and sufficient conditions on this partition to produce a valid choice of signatures? Lemma 3.7 answers this question in much the same way as Lemma 3.4 did for the signature configuration problem.

**Lemma 3.7.** For $(T = (V, E, A), \rho, \sim) \in \mathcal{T}$ and $\{u, v\} \in E$, if $\rho(\{u, v\}) \not\subseteq \hat{T}$, then $\Sigma(1_T, u) = \Sigma(1_T, v)$.

**Proof.** Suppose $\hat{T} = \{A, B\}$ and $C \in \rho(\{u, v\}) \setminus \{A, B\}$. If $\Sigma(1_T, u) \neq \Sigma(1_T, v)$, then we may form a rainbow triangle by selecting $A$ for $(1_T, u)$, $B$ for $(1_T, v)$, and $C$ for $(u, v)$.
Construction 3 (Tree Construction: $\Delta_T$). For $\mathcal{N} \subseteq \mathcal{M}^*$, fix $(M = (V, E, A), \rho, \sim_{\Sigma}) \in \mathcal{N}$ and colors $\{A, B\}$. Let $V = \cup V_i$ be the vertex partition of $M$ into weak components. Let $1_T$ be a new vertex and form $(T := (V \cup 1_T, E', A'), \rho', \sim_{\Sigma'})$ as follows:

1. $E' := E$;
2. $A' := A \cup \{(1_T) \times \{V\}\}$;
3. $\rho'(e) := \{A, B\}$ for all $e \in \{1_T\} \times \{V\}$ and $\rho' \upharpoonright E \cup A := \rho$;
4. Define $\sim_{\Sigma'}$ on $A'$ by
   - $(1_T, u) \sim_{\Sigma'} (1_T, v)$ whenever $(u, v) \in A$,
   - if $\nabla V \not\subseteq \{A, B\}$, then $(1_T, u) \sim_{\Sigma'} (1_T, v)$ for all $u \in V_i, v \in V_j$,
   - if $(u, v_1) \sim_{\Sigma'} (u, v_2)$, then $(u, v_1) \sim_{\Sigma'} (u, v_2)$ for all $u, v_i \in V$.

We write $\Delta_T(\mathcal{N})$ for the family of mixed graphs resulting from all possible choices of $(M, \rho, \sim_{\Sigma}) \in \mathcal{N}$ and possible definitions of $\sim_{\Sigma}$ in step (4).

Theorem 3.8. $T(n + 1) \subseteq \Delta_T(M(n)) \subseteq \Delta_T(M^*) \subseteq T^*$.

Proof. Since $\mathcal{M}(n) \subseteq \mathcal{M}^*$, we have $\Delta_T(\mathcal{M}(n)) \subseteq \Delta_T(\mathcal{M}^*)$. We now show that $\Delta_T(\mathcal{M}^*) \subseteq T^*$. Fix $M \in \mathcal{M}^*$, $T \in \Delta_T(M)$, and $G \in \Gamma(T)$.

By construction, it is clear that $T$ is a tree with root $1_T$ so we only need to verify $T \in T^*$, i.e., $G \in \mathcal{G}$. Since $M \in \mathcal{M}^*$, the portion of $G$ corresponding to $M$, that is $G$ with $1_T$ removed, will certainly lack rainbow triangles. Likewise, if a triangle falls entirely in the portion of $G$ corresponding to $1_T$, then it will have at most two colors. We thus only need to consider the general types of triangles presented in Figure 5.

![Figure 5](image-url)

Case (A) cannot lead to a rainbow triangle since $1_T1_T \subseteq \overline{T} = \overline{1_TU}$. Likewise, in case (B), since $1_T \rightarrow U$, the vertex in $1_T$ must be connected to $U$ by a single color. Cases (C) and (D) cannot contain a rainbow triangle since $\Sigma(1_T, U) = \Sigma(1_T, V)$.

Finally, by construction, if $\rho(U, V) \not\subseteq \overline{T}$, then $\Sigma(1_T, U) = \Sigma(1_T, V)$. Since in case (E), $\Sigma(1_T, U) \neq \Sigma(1_T, V)$ it must be the case that $\rho(U, V) \subseteq \overline{T}$ and thus the figure cannot contain a rainbow triangle. We have thus shown that $\Delta_T(\mathcal{M}_T) \subseteq T^*$.

We now show $T(n + 1) \subseteq \Delta_T(\mathcal{M}(n))$. Fix $T \in T(n + 1)$. Since $T(n + 1) \subseteq \mathcal{M}$, $T = M_k(H)$ for some $H \in \mathcal{G}_r^+$. Let $H'$ be the subgraph of $H$ induced by removing the vertices corresponding to $1_T$. Again, $H' \in \mathcal{G}_r^+$ and thus $M_k(H') \in \mathcal{M}$. By Lemma 3.6 however, $M_k(H')$ will have one less vertex than $T$ and thus $M_k(H') \in \mathcal{M}(n)$. Since $T \in \Delta_T(M_k(H')) \subseteq \Delta_T(\mathcal{M}(n))$, we have $T(n + 1) \subseteq \Delta_T(\mathcal{M}(n))$.

3.3. Proof of Theorem 3.3. We may now show that $\mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{M}^*$.
Proof. It is clear that $\mathcal{M} \subseteq \mathcal{M}'$.

Certainly $\mathcal{M}_0 \subseteq \mathcal{M}^*$ since $\mathcal{M}_0$ is just the family of Gallai graphs. Now consider $\mathcal{M}_{n+1} = \Delta_T(\mathcal{M}_n) \cup \Delta_F(\mathcal{M}_n)$ where $\mathcal{M}_n \subseteq \mathcal{M}^*$. Theorem 3.8 shows that $\Delta_T(\mathcal{M}_n) \subseteq \mathcal{M}^*$.

Fix $(M = (V, E, A), \rho, \sim) \in \Delta_F(\mathcal{M}_n)$ and $(G, \rho) \in \Gamma(M)$. Recall that $M$ was constructed by replacing vertices of $M' \in \mathcal{M}_n$ with trees $\{T_i\} \subseteq \mathcal{M}_n$ and joining these trees subject to certain constraints. Recall further that $G$ was formed by replacing the vertices of $M$, which as just mentioned can be thought of as the vertices of the collection $\{T_i\}$, with complete edge-colored multigraphs, say $\{G_i\} \in G$.

Now consider a triangle $u, v, w$ in $G$. Let $u$ fall in the portion of $G$ corresponding to $T_u$, $v$ in $T_v$, and $w$ in $T_w$. Up to relabeling, we may then assume we are in one of the following cases:

1. $T_u = T_v = T_w$,
2. $T_u, T_v, T_w$ are distinct,
3. $T_v = T_w$ and $T_u \to T_v$, or
4. $T_u = T_v$ and $T_u \to T_w$.

Cases (1) and (2) are resolved by the facts that $T_u \in \mathcal{M}^*$ and $M' \in \mathcal{M}^*$, respectively. Case (3) is straightforward since $\Sigma(T_u, T_v) = \Sigma(T_u, w) = \Sigma(T_u, v)$ and thus $\overrightarrow{uw} = \overrightarrow{vw}$.

Case (4) requires some consideration of $\Sigma(T_u, T_w)$. If it happens that $\Sigma(T_u, T_w)(u) = \Sigma(T_u, T_w)(v)$, then $\overrightarrow{uw} = \overrightarrow{vw}$ and we are done. Suppose then that $\Sigma(T_u, T_w)(u) \neq \Sigma(T_u, T_w)(v)$. By construction, this implies that $u$ and $v$ are not connected by a path in $T_u$ whose colors fall outside $\overleftarrow{T_u}$. In particular, $\overrightarrow{uw} \subseteq \overleftarrow{T_u}$. Since we also know that $\overrightarrow{uw}, \overrightarrow{vw} \subseteq \overrightarrow{T_u}$ and $|\overrightarrow{T_u}| = 2$, we are done.

\[ \square \]

4. Open Problems

We approached the topic of Gallai multigraphs in an attempt to generalize the construction of Gallai graphs for graphs lacking rainbow 4-cycles. We remain interested in this problem and in the following somewhat more general questions suggested by the multigraph perspective:

1. Can $\Delta$ be generalized to construct finite multigraphs lacking rainbow $n$-cycles for a fixed $n$?
2. Is there a construction of all finite (not necessarily complete) graphs or multigraphs lacking rainbow triangles?

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References


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