Spectral Properties of the Renormalization Group

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The renormalization group (RG) approach is largely responsible for the considerable success which has been achieved in developing a quantitative theory of phase transitions.
Physical properties emerge from considering the spectral properties of the linearization of the RG map at a fixed point. We consider real-space RG for classical lattice systems. The linearization acts on an infinite-dimensional Banach space of interactions. At a trivial fixed point (zero interaction), the spectral properties of the RG linearization can be worked out explicitly, without any approximation.

In this talk, I will talk about results obtained for the RG maps corresponding to decimation and majority rule. They indicate spectrum of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. This may serve as a lesson in what one might expect in more general situations.
Part of the talk is inspired by R.B. Israel's paper: Banach algebras and Kadanoff transformations. In particular, some of the theorems to be presented here were stated in this paper without proof.

I got this work started when I was working with my advisor Prof. William Faris in the 2008 program in Combinatorics and Statistical Mechanics at the Isaac Newton Institute in Cambridge, organized by Prof. Alan Sokal.
I greatly benefited from many useful discussions with my advisor and I especially thank him for his patience with me.
I learnt a lot about math physics from talking to the professors in my department, especially Prof. Tom Kennedy. I gained an insight into renormalization group from his carefully prepared math physics lectures.

We need some basic setup first.

- Our original and image configuration spaces are $\mathbb{Z}^d$, where d is any positive integer for decimation and $d = 1$ for majority rule.
- $\sigma : \mathbb{Z}^d \to \{-1, 1\}$, the spin function for Ising-type systems.
- σ_x, commonly referred to as the spin at site x, where $x \in \mathbb{Z}^d$ is a d-dimensional vector.
- $\sigma_X = \prod_{x \in X} \sigma_x$, and we require X to be nonempty.
- J and J' are defined on nonempty finite subsets of $\mathbb{Z}^d$ and take values in the complex numbers for the convenience of spectral theory.
- A constant pure magnetic field is such that $J(X) = 0$ except for one-point sets $\{x\}$, where $J(\{x\}) = m$, a constant.

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We consider RG transformations which are formally defined by,

$$e^{\sum_Y J'(Y)\sigma'_Y} = \sum_{\sigma} T(\sigma, \sigma') e^{\sum_X J(X)\sigma_X} \tag{1}$$

where T is a probability kernel from the original configuration space to the image configuration space, with $\sum_{\sigma'} T(\sigma, \sigma') = 1$ for every σ.

We work in two types of Banach spaces, $\mathcal{B}$ (with pair space $\mathcal{B}^*$) and $\mathcal{B}_r$ (with pair space $\mathcal{B}_r^*$).
$\mathcal{B}$ ($\mathcal{B}_r$) is technically not the dual space of $\mathcal{B}^*$ ($\mathcal{B}_r^*$) and $\mathcal{B}^*$ ($\mathcal{B}_r^*$) is technically not the dual space of $\mathcal{B}$ ($\mathcal{B}_r$). $\mathcal{B}$ and $\mathcal{B}^*$ ($\mathcal{B}_r$ and $\mathcal{B}_r^*$) are paired in the sense that each one is part of the dual space of another, or in other words, each one consists of continuous linear functions defined on another.

The norms are respectively defined by,

$$||J|| = \sup_{x \in \mathcal{L}} \sum_{x \in X} |J(X)| \tag{2}$$

$$||J||^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X)| \tag{3}$$

$$||J||_r = \sup_{x \in \mathcal{L}} \sum_{x \in X} |J(X)| e^{r|X|} \tag{4}$$

$$||J||_r^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X)| e^{-r|X|} \tag{5}$$

where the constant $r > 0$ and $|X|$ is the cardinality of the set X. It is not hard to verify that these are suitable definitions.

Note: Here the $||J||_r$ and the $||J||_r^*$ norms are somewhat different from the corresponding norms defined in Israel's paper. Without much difficulty, we can verify that same results to be presented here would also hold if we adopt his definitions.

For decimation transformation in a d-dimensional space with blocking factor b^d, the probability kernel $T(\sigma, \sigma')$ is defined by

$$T(\sigma, \sigma') = \begin{cases} 1 & \text{if } \sigma'_x = \sigma_{b^d x} \text{ for all } x; \\ 0 & \text{otherwise.} \end{cases}$$

where the slightly strange notation $b^d x$ denotes the d-dimensional vector obtained by multiplying every component of the d-dimensional vector x by b.

At the trivial fixed point (zero interaction), with the help of Fourier theory on the group $\{-1,1\}^{\mathbb{Z}^d}$, we can work out the explicit infinite volume expression of the linearization of the RG map ('**L**' stands for linearization),

$$\frac{\partial J'(X)}{\partial J(Y)} = \delta(Y, b^d X) \tag{6}$$

$$\mathbf{L}J(X) = J(b^d X) \tag{7}$$

where $b^d X = \cup_{x \in X}\{b^d x\}$.

By the usual correspondence between adjoint operators,

$$\sum_X J_1(X)\mathbf{L}J_2(X) = \sum_Y J_2(Y)\mathbf{L}^* J_1(Y)$$

The explicit expression of the adjoint operator $\mathbf{L}^*$ is also obtained,

$$\mathbf{L}^* J(X) = \begin{cases} J(\frac{1}{b^d}X) & \text{if } X \text{ is a block;} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

where X is a block means that $X = b^d Y$ for some set Y, and we define $\frac{1}{b^d}X = Y$.

Theorem (Israel)

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Proposition

$||\mathbf{L}|| = 1$.

Proof.

We check that for each fixed $x \in \mathcal{L}$, $\sum_{x \in X} |\mathbf{L}J(X)| \leq ||J||$, which would imply $||\mathbf{L}|| \leq 1$.
Recall that for the eigenvector J, $\mathbf{L}J(X) = J(b^d X)$, so naturally,

$$\sum_{x \in X} |\mathbf{L}J(X)| = \sum_{x \in X} |J(b^d X)| = \sum_{b^d x \in b^d X} |J(b^d X)|$$
$$\leq \sum_{b^d x \in X} |J(X)| \leq ||J||$$

Realizing that a constant pure magnetic field is an eigenvector with eigenvalue 1, we conclude $||\mathbf{L}|| = 1$. □

Corollary

Every eigenvalue λ of $\mathbf{L}$ satisfies $|\lambda| \leq 1$.

Proposition

Every $|\lambda| \leq 1$ is an eigenvalue.

Proof.

For a generic λ, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each λ.
The eigenvector J is defined by,

$$J(\{(b^n, 0, ..., 0)\}) = \lambda^n J(\{(1, 0, ..., 0)\}) = \lambda^n$$

and for all the other subsets X, $J(X)$ is set to zero. □

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Proposition

$||\mathbf{L}^*|| = 1.$

Proof.

Standard analysis fact for Banach spaces, as $||\mathbf{L}|| = 1$. □

Proposition

For every eigenvalue $\lambda \neq 0$, there is no nontrivial eigenvector.

Proof.

Fix an arbitrary finite subset X of the infinite lattice, after a finite number of iterations of $\mathbf{L}^*$ (say n times), X will not take on the shape of a block.
Thus $(\mathbf{L}^*)^{n+1} J(X) = 0 = \lambda^{n+1} J(X)$, which implies $J(X) = 0$. □

Proposition

For $\lambda = 0$, there is no nontrivial eigenvector.

Proof.

Suppose the nontrivial eigenvector $J(X) = m \neq 0$ for some finite subset X, then the crucial fact that we can always find Y, with $\mathbf{L}^* J(Y) = J(X)$ will do the job.
$\mathbf{L}^* J(Y) = \lambda J(Y) = 0$, we reach a contradiction. □

Corollary

The point spectrum of $\mathbf{L}^*$ is empty.

In order to verify our theorem, the only thing left to show now is that for $|\lambda| \leq 1$, $\overline{\text{Range}(\lambda I - \mathbf{L}^*)} \neq \mathcal{B}^*$.
We divide into two cases: $|\lambda| < 1$ and $|\lambda| = 1$.

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We divide into two cases: $|\lambda| < 1$ and $|\lambda| = 1$.

- $|\lambda| < 1$.

Define $J(\{(b^n, 0, ..., 0)\}) = \overline{\lambda}^n$, and $J(X) = 0$ for all other subsets X.

Recall the norm definition in $\mathcal{B}^*$, $||J||^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X)|$.

In our current case,

$$||J||^* = \sum_{n=0}^{\infty} |\overline{\lambda}|^n = \frac{1}{1 - |\lambda|} < \infty$$

This says that J lies in $\mathcal{B}^*$.

However, J can not be approximated by any J' in Range$(\lambda I - \mathbf{L}^*)$.

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However, J can not be approximated by any J' in Range$(\lambda I - \mathbf{L}^*)$.

To see this, note that

$$J'(X) = \begin{cases} \lambda K(X) - K(\frac{1}{b^d}X), & \text{if } X \text{ is a block} \\ \lambda K(X), & \text{otherwise} \end{cases}$$

for some K that lies in $\mathcal{B}^*$.
Also

$$\begin{aligned} ||J - J'||^* &= \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X) - J'(X)| \\ &\geq \sum_{x=(b^n,0,\ldots,0)} \sup_{x \in X} \frac{1}{|X|} |J(X) - J'(X)| \\ &\geq \sum_{n=0}^{\infty} |J(\{(b^n, 0, \ldots, 0)\}) - J'(\{(b^n, 0, \ldots, 0)\})| \end{aligned}$$

However,

$$\sum_{n=0}^{\infty} \overline{J(\{(b^n,0,...,0)\})} J'(\{(b^n,0,...,0)\}) = \sum_{n=0}^{\infty} \lambda^n J'(\{(b^n,0,...,0)\})$$

$$= \lambda K(\{(1,0,...,0)\}) + \lambda(\lambda K(\{(b^n,0,...,0)\}) - K(\{(1,0,...,0)\})) + \cdots$$

$$= 0$$

$$\sum_{n=0}^{\infty} |J(\{(b^n,0,...,0)\}) - J'(\{(b^n,0,...,0)\})|$$

$$\geq \sqrt{\sum_{n=0}^{\infty} |J(\{(b^n,0,...,0)\}) - J'(\{(b^n,0,...,0)\})|^2}$$

$$\geq \sqrt{\sum_{n=0}^{\infty} |\overline{\lambda}^n|^2} = \sqrt{\frac{1}{1-|\lambda|^2}}$$

- $|\lambda| = 1$.

Define $J(\{(1, 0, ..., 0)\}) = 1$, and $J(X) = 0$ for all other subsets X. Suppose there exists a J' approximating J such that

$$\frac{1}{2} \geq ||J - J'||^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X) - J'(X)|$$

$$\geq |\lambda K(\{(1, 0, ..., 0)\}) - 1| + |\lambda K(\{(b^n, 0, ..., 0)\}) - K(\{(1, 0, ..., 0)\})| + \cdot$$

where K is defined as before.

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where K is defined as before.

Then, as $|\lambda| = 1$, we would have, for any $n \geq 0$.

$$|\lambda^{n+1} K(\{(b^n, 0, ..., 0)\}) - 1| \leq \frac{1}{2} \qquad |K(\{(b^n, 0, ..., 0)\})| \geq \frac{1}{2}$$

But then,

$$||K||^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |K(X)| \geq \sum_{n=0}^{\infty} |K(\{(b^n, 0, ..., 0)\})| = \infty$$

contradiction.

Theorem

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These theorems might not come as a surprise once we realize the similarity between $\mathbf{L}/\mathbf{L}^*$ and left translation/right translation: $\mathbf{L}$ acts like left translation and $\mathbf{L}^*$ acts like right translation on sequences $(X, b^d X, ...)$ for all possible subsets X.
Moreover, ignoring multiplicity of the eigenvalues, the spectrum of $\mathbf{L}$ is the same as that of left translation in l^∞, and the spectrum of $\mathbf{L}^*$ is the same as that of right translation in l^1, which might be a result of the norms in our Banach spaces being a combination of l^∞ and l^1 norms.

For majority rule transformation in one-dimension with blocking factor 3, the probability kernel $T(\sigma, \sigma')$ is defined by

$$T(\sigma,\sigma') = \begin{cases} 1 & \text{if } \sigma'_x = \frac{1}{2}(\sigma_{3x-1} + \sigma_{3x} + \sigma_{3x+1} - \sigma_{3x-1}\sigma_{3x}\sigma_{3x+1}),\ \forall x; \\ 0 & \text{otherwise.} \end{cases}$$

At the trivial fixed point (zero interaction), with the help of Fourier theory on $\{-1,1\}^{\mathbb{Z}}$, we can work out the explicit infinite volume expression of the linearization of the RG map ('**L**' stands for linearization),

$$\frac{\partial J'(X)}{\partial J(Y)} = \begin{cases} \prod_{x\in X} \chi(Y \cap \mathbf{R}x) & \text{if } Y \subset \mathbf{R}X; \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

$$\mathbf{L}J(X) = \sum_{Y: Y\subset \mathbf{R}X} J(Y) \prod_{x\in X} \chi(Y \cap \mathbf{R}x) \tag{10}$$

where χ is defined by,

$$\chi(X) = \begin{cases} \frac{1}{2} & \text{if } |X| = 1; \\ -\frac{1}{2} & \text{if } |X| = 3; \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbf{R}x$ denotes the set $\{3x-1, 3x, 3x+1\}$ and $\mathbf{R}X$ denotes the set $\cup_{x\in X}\{3x-1, 3x, 3x+1\}$. ('**R**' stands for rescaling)

By the usual correspondence between adjoint operators,

$$\sum_X J_1(X)\mathbf{L}J_2(X) = \sum_Y J_2(Y)\mathbf{L}^* J_1(Y)$$

The explicit expression of the adjoint operator $\mathbf{L}^*$ is also obtained,

$$\mathbf{L}^* J(X) = \begin{cases} \pm\frac{1}{2^n} J(\bigcup_n \{m\}) & \text{if } X = \bigcup_n W_m; \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

where W_m is either one of the following four sets, $\{3m-1\}, \{3m\}, \{3m+1\}$ or $\{3m-1, 3m, 3m+1\}$ (m integer). $\bigcup_n W_m$ is a finite union of n of these disjoint W_m's. And "+" is taken when $|X| - n$ is divisible by 4; and "−" is taken otherwise.

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Proposition

$||\mathbf{L}|| = \frac{3}{2}$.

Proof.

We check that for each fixed $x \in \mathcal{L}$, $\sum_{x \in X} |\mathbf{L}J(X)| \leq \frac{3}{2}||J||$, which would imply $||\mathbf{L}|| \leq \frac{3}{2}$.
As $x \in X$, $\mathbf{L}J(X)$ is a linear combination of $J(Y)$'s, $|Y \cap \mathbf{R}x|$ being either 1 or 3. Due to the definition of χ, the coefficients of $J(Y)$'s are bounded above by $\frac{1}{2}$.
We group the sum $\mathbf{L}J(X)$ into three parts:

- $Y \cap \mathbf{R}x = \{3x - 1\}$
- $Y \cap \mathbf{R}x = \{3x\}$ or $Y \cap \mathbf{R}x = \{3x - 1, 3x, 3x + 1\}$
- $Y \cap \mathbf{R}x = \{3x + 1\}$

Proof.

Ignoring the coefficients of $J(Y)$'s, these three parts are less than or equal to $\sum_{3x-1\in X} |J(X)|$, $\sum_{3x\in X} |J(X)|$, and $\sum_{3x+1\in X} |J(X)|$ respectively, therefore bounded above by $||J||$ by definition. Realizing that a constant pure magnetic field is an eigenvector with eigenvalue $\frac{3}{2}$, we conclude $||\mathbf{L}|| = \frac{3}{2}$. □

Corollary

Every eigenvalue λ *of* $\mathbf{L}$ *satisfies* $|\lambda| \leq \frac{3}{2}$.

Proposition

Every $|\lambda| \leq \frac{3}{2}$ *is an eigenvalue.*

Proof.

For a generic λ, we display one eigenvector here. In fact, with some further thought, it is not hard to show that there are infinitely many eigenvectors for each λ.
The eigenvector J is defined by,

$$J(\{0\}) = J(\{1\})$$

$$J(\{-1\}) = 2(\lambda - 1)J(\{0\})$$

$$J(\{3n-1\}) = J(\{3n\}) = J(\{3n+1\}) = \frac{2}{3}\lambda J(\{n\}) \text{ for } n \neq 0$$

and for all the other subsets X, $J(X)$ is set to zero. □

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Proposition

For every eigenvalue $\lambda \neq 0$ and $\lambda \neq \frac{1}{2}$, there is no nontrivial eigenvector.

Proof.

Fix an arbitrary finite subset X of the infinite lattice. For $\lambda \neq 0$, $J(X)$ is either zero or a nonzero constant multiple of $J(\{0\})$ as a result of the action of $\mathbf{L}^*$.
For $\lambda \neq \frac{1}{2}$, $\lambda J(\{0\}) = \mathbf{L}^* J(\{0\}) = \frac{1}{2} J(\{0\})$, which implies that $J(\{0\}) = 0$. □

Proposition

For $\lambda = 0$, there is no nontrivial eigenvector.

Proof.

Suppose the nontrivial eigenvector $J(X) = m \neq 0$ for some finite subset X, then the crucial fact that we can always find Y, with $\mathbf{L}^* J(Y)$ a nonzero constant multiple of $J(X)$ will do the job. $\mathbf{L}^* J(Y) = \lambda J(Y) = 0$, we reach a contradiction. □

Proposition

For $\lambda = \frac{1}{2}$, every nontrivial eigenvector has norm infinity.

Proof.

We must have $J(\{0\}) = m \neq 0$ in order for J to be nontrivial.
As $\frac{1}{2}J(\{-1\}) = \mathbf{L}^* J(\{-1\}) = \frac{1}{2}J(\{0\})$, we see that $J(\{-1\}) = m$ also.
Following similar fashion, $J(\{n\}) = m$ for arbitrary n.
But then,

$$||J||^* = \sum_{x \in \mathcal{L}} \sup_{x \in X} \frac{1}{|X|} |J(X)| = \infty$$

contradiction. □

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