Critical phenomena in exponential random graphs

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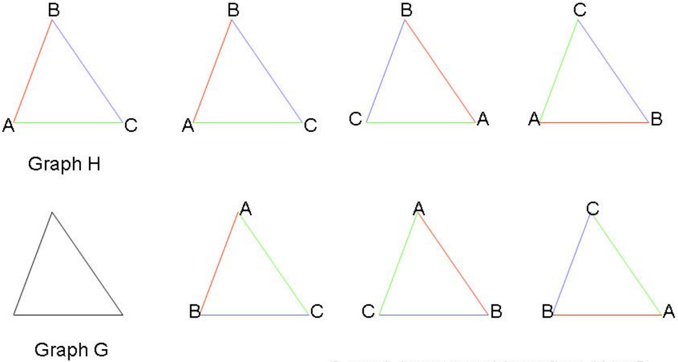
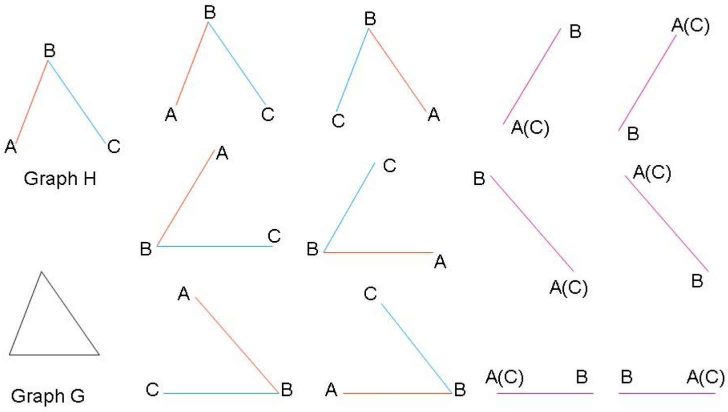


6 graph homomorphisms from H to G



12 graph homomorphisms from H to G

Probability space: The set $\mathcal{G}_n$ of all simple graphs G_n on n vertices.
Probability mass function:

$$\mathbb{P}_n^{\beta}(G_n) = \exp\left(n^2(\beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n) - \psi_n^{\beta})\right).$$

H_1 is a single edge.
Graph homomorphism density $t(H_i, G_n) = \frac{|\hom(H_i, G_n)|}{|V(G_n)|^{|V(H_i)|}}$.
Normalization constant (free energy) ψ_n^{β}:

$$\psi_n^{\beta} = \frac{1}{n^2} \log \sum_{G_n \in \mathcal{G}_n} \exp\left(n^2\left(\beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n)\right)\right).$$

Recent survey: Fienberg, Introduction to papers on the modeling and analysis of network data I & II. arXiv: 1010.3882 & 1011.1717.

$k = 1$:

$$\begin{aligned} \mathbb{P}_n^{\beta}(G_n) &= \exp\left(n^2(\beta_1 t(H_1, G_n) - \psi_n^{\beta})\right) \\ &= \exp\left(2\beta_1 E(G_n) - n^2 \psi_n^{\beta}\right). \end{aligned}$$

Erdős-Rényi graph $G(n, \rho)$,

$$\mathbb{P}_n^{\rho}(G_n) = \rho^{E(G_n)}(1-\rho)^{\binom{n}{2} - E(G_n)}.$$

Include edges independently with parameter $\rho = e^{2\beta_1}/(1 + e^{2\beta_1})$.

$$\exp(n^2\psi_n^{\beta}) = \sum_{G_n \in \mathcal{G}_n} \exp\left(2\beta_1 E(G_n)\right) = \left(\frac{1}{1-\rho}\right)^{\binom{n}{2}}.$$

Theory of graph limits: Graphon space $\mathcal{W}$ is the space of all symmetric measurable functions from $[0,1]^2$ into $[0,1]$.
A sequence of graphs $\{G_n\}_{n\geq 1}$ is said to converge to $h \in \mathcal{W}$ if for every finite simple graph H,

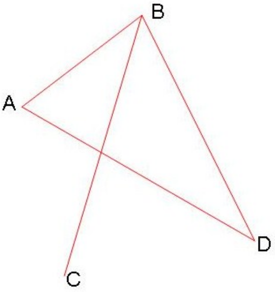
$$\lim t(H, G_n) = t(H, h)$$

$$= \int_{[0,1]^k} \prod_{(i,j)\in E(H)} h(x_i, x_j) dx_1 \ldots dx_k.$$

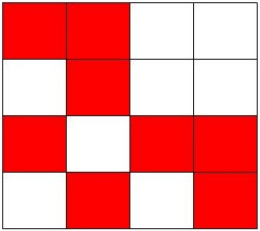
(Lovász and coauthors)
Intuition: The interval $[0,1]$ represents a 'continuum' of vertices, and $h(x, y)$ denotes the probability of putting an edge between x and y.

Example: Erdős-Rényi graph $G(n, \rho)$, $h(x, y) = \rho$.
Example: Any $G \in \mathcal{G}_n$,

$$f^G(x, y) = \begin{cases} 1, & \text{if } (\lceil nx, ny \rceil) \text{ is an edge in } G; \\ 0, & \text{otherwise.} \end{cases}$$



Graph G



Graphon f^G

Large deviation:

$$\psi_\infty^\beta = \sup_{h \in \mathcal{W}} \left(\beta_1 t(H_1, h) + \cdots + \beta_k t(H_k, h) - \frac{1}{2} \iint_{[0,1]^2} I(h) dxdy \right),$$

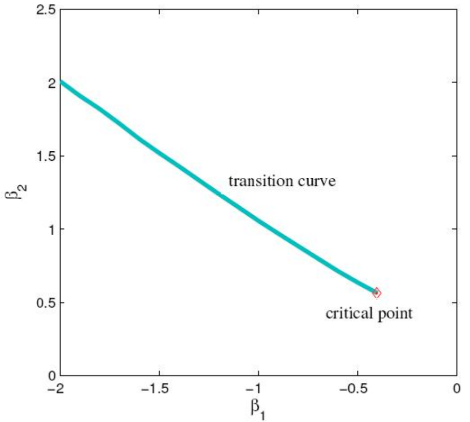
where $I : [0,1] \to \mathbb{R}$ is the function

$$I(u) = \frac{1}{2} u \log u + \frac{1}{2}(1-u) \log(1-u).$$

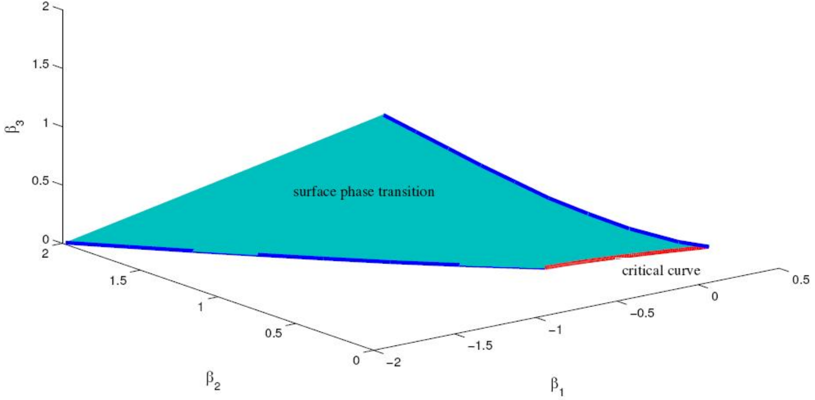
$k \geq 2$, $\beta_2, ..., \beta_k$ nonnegative ('attractive'): G_n behaves like the Erdős-Rényi graph $G(n, u^*)$ in the large n limit, where $u^* \in [0,1]$ maximizes

$$l(u) = \beta_1 u^{E(H_1)} + \cdots + \beta_k u^{E(H_k)} - \frac{1}{2} u \log u - \frac{1}{2}(1-u)\log(1-u).$$

(Chatterjee and Diaconis, Estimating and understanding exponential random graph models. arXiv: 1102.2650.)



The phase transition curve in the (β_1, β_2) plane. H_1 is a single edge and H_2 has 3 edges.
(Radin and Y, Phase transitions in exponential random graphs. arXiv: 1108.0649.)

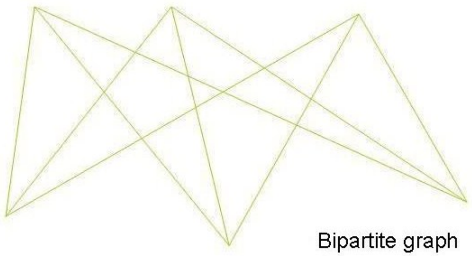


The phase transition surface in the $(\beta_1, \beta_2, \beta_3)$ space. H_1 is a single edge, H_2 has 3 edges, and H_3 has 5 edges.
(Y, Critical phenomena in exponential random graphs. arXiv: 1208.2992.)

Conjecture: The parameter space of the ‘attractive’ model consists of a single phase with first order phase transition(s) across one (or more) hypersurfaces and second order phase transition(s) along their boundaries.

Universality: All hypersurfaces asymptotically approach a common hyperplane $\sum_{i=1}^{k} \beta_i = 0$.

$k = 2$, β_2 large negative ('repulsive'): G_n looks like a complete $(\chi(H_2) - 1)$-equipartite graph.



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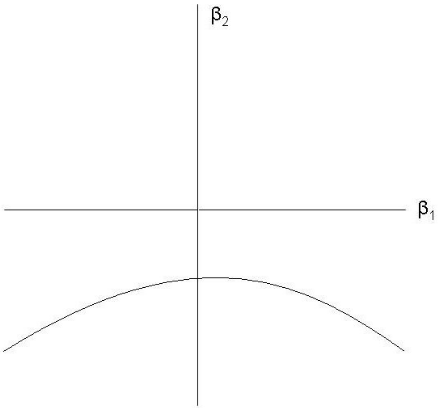
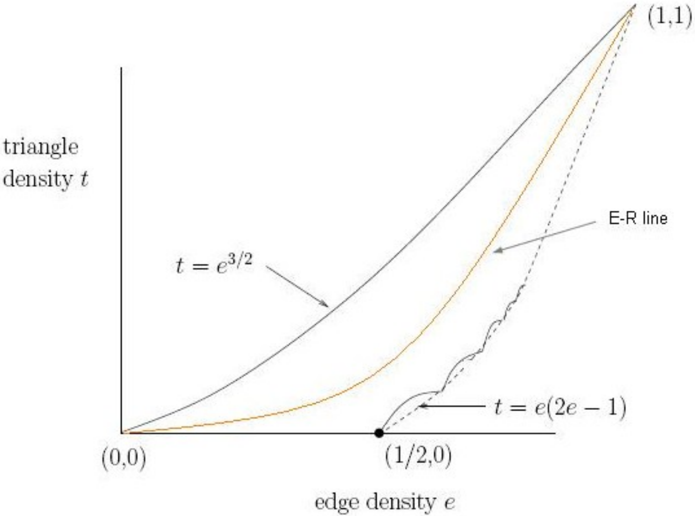


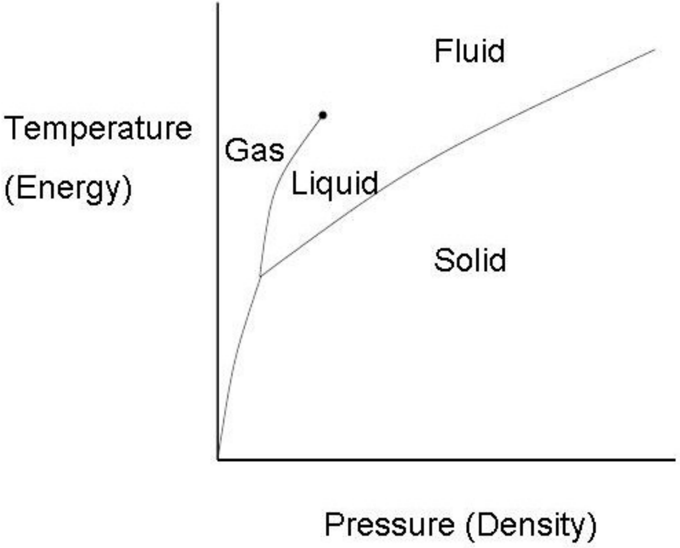
(Aristoff and Radin, Emergent structures in large networks. arXiv: 1110.1912.
Y, http://www.ma.utexas.edu/users/myin/Talk.pdf.)


(1,1)
triangle
density t
E-R line
$t = e^{3/2}$
$t = e(2e-1)$
(0,0)
(1/2,0)
edge density e

Question: In the large n limit, is every combination of edge density and triangle density obtainable?

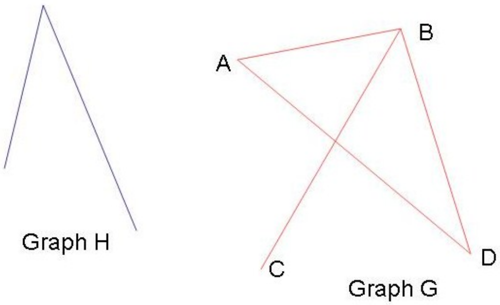
- Above E-R line?
- Fix β_1. What happens when $\beta_2 \to -\infty$?
- Fix α. Let $\beta_1 = \alpha\beta_2$. What happens when $\beta_2 \to -\infty$?
- Scallops? Or more generally?

Euler-Lagrange: $\beta_1 + 3\beta_2 \int_0^1 h(x,z)h(y,z)dz = I'(h(x,y))$.


Fluid
Temperature
(Energy)
Gas
Liquid
Solid
Pressure (Density)

Alternating star model: H_i is an i-star, and $\beta_i = (-1)^{i-1}\beta^{-i}$ with $\beta > 0$ a constant.
Exhibit the desired transitivity and clumping properties in simulations.
(Snijders, Pattison, Robins, and Handcock, New specifications for exponential random graph models.
http://www.csss.washington.edu/Papers/wp42.pdf.)

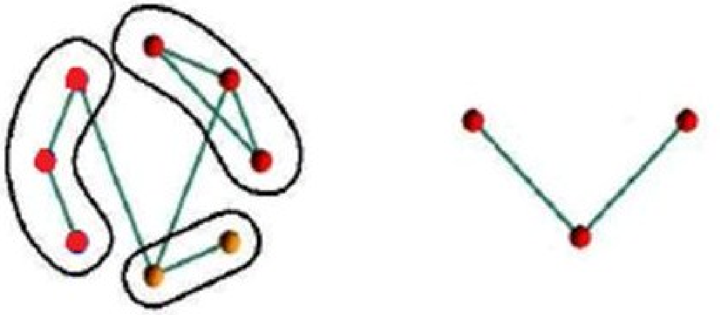
Lattice gas representation: $t(H, G_n) = \sum d(H, X)\sigma_X$.
Exact graph homomorphism density $d(H, X) = 2/4^3$ for X an edge or a 2-star.



Suppose $H \in \mathcal{G}_m$, then $t_{ij}(H, G_n)$, the sum of the exact homomorphism densities $d(H, X)$ with $(i, j) \in X$, has an upper bound $\frac{m(m-1)}{n^2}$.

Any exponential random graph model may be viewed as a lattice gas model with a finite Banach space norm.
$\sum_{i=1}^{k} |\beta_i|$ small: Convergent power series expansion (high-temperature expansion) for the limiting free energy. No phase transition.
(Y, A cluster expansion approach to exponential random graph models. arXiv: 1202.5587.)

Renormalization (graph perspective):



(Song, Havlin, and Makse, Self-similarity of complex networks. arXiv: cond-mat/0503078.)

Renormalization (lattice gas perspective):

$$\exp(-\bar{H}(\bar{\sigma})) = \sum_{\sigma} T(\sigma, \bar{\sigma}) \exp(-H(\sigma)).$$

$$\begin{array}{ccc} \text{original random graph} & \overset{\text{(I)}}{\longrightarrow} & \text{renormalized random graph} \\ \downarrow_{\text{(II)}} & & \downarrow_{\text{(IV)}} \\ \text{original lattice gas} & \overset{\text{(III)}}{\longrightarrow} & \text{renormalized lattice gas} \end{array}$$

Thank You!

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