Random generators of given orders and the smallest simple Moufang loop

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ABSTRACT. The probability that m randomly chosen elements of a finite power associative loop C have prescribed orders and generate C is calculated in terms of certain constants related to the action of Aut(C) on the subloop lattice of C. As an illustration, all meaningful probabilities of random generation by elements of given orders are found for the smallest nonassociative simple Moufang loop.

1. Random generators of given orders

Let C be a power associative loop, i.e., a loop where every element generates a group. Given an m-tuple \(a = (a_0, \ldots, a_{m-1})\) of elements of C, let \(A = \{A_i\}_{i=0}^m\) be the sequence of nested subloops \(A_i \leq C\) such that \(A_0\) is the smallest subloop of \(C\), and \(A_{i+1} = \langle A_i, a_i \rangle\). Note that \(A_m\) is independent of the order of the elements \(a_0, \ldots, a_{m-1}\) in \(a\).

Denote by \(\text{Gen}_m(C)\) the set of all m-tuples \(a \in C^m\) with \(A_m = C\). Then the probability that \(m\) randomly chosen elements of \(C\) generate \(C\) is

\[
p_m(C) = |C|^{-m} \cdot |\text{Gen}_m(C)|.
\]

This notion can be refined in a natural way. For \(1 \leq i \leq n\), let \(D_i\) be the set of all elements of \(C\) of order \(i\). Two \(m\)-tuples of integers \(r = (r_0, \ldots, r_{m-1})\), \(s = (s_0, \ldots, s_{m-1})\) are said to be of the same type if \(r_0, \ldots, r_{m-1}\) is a permutation of \(s_0, \ldots, s_{m-1}\). We say that \(a = (a_0, \ldots, a_{m-1}) \in C^m\) is of type \(r\) if there is \(s = (s_0, \ldots, s_{m-1})\) of the same type as \(r\) satisfying \(a_i \in D_{s_i}\), for \(0 \leq i \leq m - 1\).

Let \(\text{Gen}_r(C) \subseteq \text{Gen}_m(C)\) be the set of generating \(m\)-tuples of type \(r\). Then

\[
p_r(C) = |C|^{-m} \cdot |\text{Gen}_r(C)|
\]

is the probability that \(m\) randomly chosen elements \(a_0, \ldots, a_{m-1} \in C\) generate \(C\) and \((a_0, \ldots, a_{m-1})\) is of type \(r\).

Presented by E. W. Kiss.
Received October 4, 2001; accepted in final form February 10, 2003.
2000 Mathematics Subject Classification: 20N05, 20F05, 06B99.
Key words and phrases: Random generators of given orders, Moufang loops, Paige loops.
While working on this paper the author has been partially supported by Grant Agency of Charles University, grant number 269/2001/B-MAT/MFF.
For $A, B \leq C$ and an integer $i$, let $\Gamma_i(A, B)$ be the cardinality of the set of all elements $x \in D_i$ such that $\langle A, x \rangle \in O_B$, where $O_B$ is the orbit of $B$ under the natural action of $\text{Aut}(C)$ on the subloop lattice of $C$. Also, let $\Gamma(A, B)$ be the cardinality of the set of all elements $x \in C$ such that $\langle A, x \rangle \in O_B$.

We are going to divide $C^m$ into certain equivalence classes. Two $m$-tuples $a, b \in C^m$ with associated nested subloops $\{A_i\}_{i=0}^{m}, \{B_i\}_{i=0}^{m}$ will be called orbit-equivalent if $A_i \in O_{B_i}$, for $0 \leq i \leq m$. We write $a \sim b$.

The size of the equivalence class $a_\sim$ is easy to calculate with help of the constants $\langle A; B \rangle$. There are $\langle A_0; A_1 \rangle$ elements $x$ such that $\langle A_0, x \rangle \in O_{A_1}$. Once we are in the orbit $O_{A_i}$, we can continue on the way to $O_{A_{i+1}}$ by adding one of the $\langle A_i; A_{i+1} \rangle$ elements $x_{i+1}$ to $\langle x_0, \ldots, x_i \rangle$. Thus,

$$|a_\sim| = \prod_{i=0}^{m-1} \Gamma(A_i, A_{i+1}). \quad (3)$$

Since

$$|\text{Gen}_m(C)| = \sum_{a_\sim \in \text{Gen}_m(C)/\sim} |a_\sim|, \quad (4)$$

we can combine (1), (3) and (4) to obtain

$$p_m(C) = |C|^{-m} \sum_{a_\sim \in \text{Gen}_m(C)/\sim} \prod_{i=0}^{m-1} \Gamma(A_i, A_{i+1}). \quad (5)$$

**Example 1.1.** Let us illustrate (5) by calculating the probability that two randomly chosen elements of $S_3$ generate the entire symmetric group. Let $e$ be the neutral element of $S_3$. There are $3$ subgroups isomorphic to $C_2$ (all in one orbit of transitivity) and a unique subgroup isomorphic to $C_3$. As $\Gamma(e, C_2) = 3$, $\Gamma(e, C_3) = 2$, $\Gamma(C_2, S_3) = 4$ and $\Gamma(C_3, S_3) = 3$, we have $p_2(S_3) = (3 \cdot 4 + 2 \cdot 3)/36 = 1/2$, as expected.

Write $\sim_r$ for the restriction of $\sim$ onto $\text{Gen}_r(C)$, and observe that

$$|a_{\sim_r}| = \sum_{s=(s_0, \ldots, s_{m-1})} \prod_{i=0}^{m-1} \Gamma_{s_i}(A_i, A_{i+1}), \quad (6)$$

where the summation runs over all $m$-tuples $s$ of the same type as $r$. Consequently,

$$p_r(C) = |C|^{-m} \sum_{a_{\sim_r} \in \text{Gen}_r(C)/\sim_r} \sum_{s=(s_0, \ldots, s_{m-1})} \prod_{i=0}^{m-1} \Gamma_{s_i}(A_i, A_{i+1}). \quad (7)$$

**Remark 1.2.** All concepts of this section can be generalized to any finite universal algebra $C$ with subsets $D_i$ closed under the action of $\text{Aut}(C)$.
2. Random generators of given orders for $M^*(2)$

We assume from now on that the reader is familiar with the notation and terminology of [2].

The value of $\Gamma_1(A, B)$ in (7) can be calculated with help of Hasse constants, provided $A$ is maximal in $B$. Namely,

$$\Gamma_1(A, B) = H_C^+(A|B) \cdot |D_i \cap (B \setminus A)|.$$  \hfill (8)

This is obvious because $H_C^+(A|B)$ counts the number of subloops of $C$ containing $A$ and in the same orbit as $B$.

By (7) and (8), we should be able to calculate $p_m(C)$, $p_v(C)$ when all Hasse constants for $C$ are known and when the lattice of subloops of $C$ is not too high. These probabilities can be used alongside order statistics to recognize black box loops, for instance (cf. [1]).

Building on the results of [2] substantially, we proceed to calculate all meaningful probabilities $p_m(C)$, $p_v(C)$ for $C = M^*(2)$, the smallest nonassociative simple Moufang loop.

All Hasse constants for $C$ are summarized in [2, Fig. 1], so we can easily evaluate all constants $\Gamma_1(A, B)$ with $A$ maximal in $B$. For example, since $H_C^+(E_8|M(A_4)) = 3$ and $|D_2 \cap (M(A_4) \setminus E_8)| = 8$, we have $\Gamma_2(E_8, M(A_4)) = 24$.

Apart from trivialities, the remaining constants to be calculated are

$$\Gamma_1(C_2, A_4), \quad \Gamma_1(S_3, C), \quad \Gamma_1(A_4, C), \quad \Gamma_1(E_8^+, C),$$

$$\Gamma_1(E_8^-), M(A_4), \quad \Gamma_1(E_8^+, M(A_4)), \quad \Gamma_1(E_8^+, C), \quad \Gamma_1(E_8, C),$$

for $i = 2, 3$. As some invention is needed here, we show how to obtain all of them.

We begin with $\Gamma_i(S_3, C)$. Let $G$ be a copy of $S_3$ in $C$. For any element $x \not\in G$, we must have $(G, x) \cong M(S_3), M(A_4)$, or $C$. Therefore, $\Gamma_i(S_3, C) = (i-1) \cdot H_C(C_i) - \Gamma_i(S_3, M(S_3)) - \Gamma_i(S_3, M(A_4)) - (i-1) \cdot H_{S_3}(C_i)$, for $i = 2, 3$. Consequently,

$$\Gamma_2(S_3, C) = 63 - 6 - 36 - 3 = 18, \quad \Gamma_3(S_3, C) = 56 - 0 - 18 - 2 = 36.$$

Similarly,

$$\Gamma_2(C_2, A_4) = 0, \quad \Gamma_3(C_2, A_4) = 24,$$

$$\Gamma_2(A_4, C) = 48, \quad \Gamma_3(A_4, C) = 48,$$

$$\Gamma_2(E_8, C) = 32, \quad \Gamma_3(E_8, C) = 32.$$

A more detailed analysis of the subloop lattice of $C$ allows us to calculate the remaining eight constants.

**Lemma 2.1.** Let $G \in O^-$, and let $M_1, M_2, M_3$ be the three copies of $M(A_4)$ containing $G$. Then $M_i \cap M_j$ contains no element of order 3, for $i \neq j$, and
$M_1 \cap M_2 \cap M_3$ is the unique copy of $E_8$ containing $G$. In particular, 
\[ \Gamma_3(E_4^+, M(A_4)) = 24, \quad \Gamma_3(E_4^-, C) = 24, \quad \Gamma_2(E_4^+, M(A_4)) = 24, \quad \Gamma_2(E_4^-, C) = 8. \]

Proof. Assume there is $x \in M_i \cap M_j$, $|x| = 3$, for some $i \neq j$. Then $M_i = \langle G, x \rangle = M_j$, because $\mathcal{H}_C(E_4^-|A_4) = 0$, a contradiction. Thus $M_1 \cup M_2 \cup M_3$ contains $3 \cdot 8 = 24$ elements $x$ of order 3 such that $\langle G, x \rangle \in O_{M(A_4)}$.

Let $H$ be the unique copy of $E_8$ containing $G$. We must have $H = M_1 \cap M_2 \cap M_3$, since $\mathcal{H}_C(E_8|M(A_4)) = 3$. Therefore $M_1 \cup M_2 \cup M_3$ contains $3 \cdot (12 - 4) = 24$ involutions $x$ such that $\langle G, x \rangle \in O_{M(A_4)}$. The constants $\Gamma_i(E_4^+, C)$ are then easy to calculate with help of Figure 1. \qed

It is conceivable that there is $G \in O^+$ and $x \in C$ such that $\langle G, x \rangle = C$. It is not so, though.

Lemma 2.2. In $C$, we have 
\[ \Gamma_3(E_4^+, M(A_4)) = 48, \quad \Gamma_2(E_4^+, M(A_4)) = 48, \quad \Gamma_3(E_4^+, C) = 0, \quad \Gamma_2(E_4^+, C) = 0. \]

Proof. Pick $G \in O^+$, and let $M_1, \ldots, M_7$ be the seven copies of $M(A_4)$ containing $G$. We claim that $(M_i \cap M_j)^2 = e$, for $i \neq j$. Assume it is not true, and let $x$ be an element of order 3 contained in $M_i \cap M_j$. Then $A_4 \cong \langle G, x \rangle \leq M_i \cap M_j$ shows that $\mathcal{H}_{M(A_4)}(A_4) \geq 2$, a contradiction. Thus $\bigcup_{i=1}^7 M_i$ contains all $8 \cdot 7 = 56$ elements of order 3. In particular, we have $\langle G, x \rangle \neq C$ for any element $x$ of order 3. This translates into
\[ \Gamma_3(E_4^+, C) = 0, \quad \Gamma_3(E_4^+, M(A_4)) = 56 - \Gamma_3(E_4^+, A_4) = 48. \]

We proceed carefully to show that $\Gamma_2(E_4^+, C) = 0$. The group $G$ is contained in a single copy $A$ of $A_4$, that is in turn contained in a single copy $M_1$ of $M(A_4)$. Let $H_1, H_2, H_3 \leq M_1$ be the three copies of $E_8$ containing $G$ (see the proof of Lemma 13.1 [2]). Observe that $H_1 \cup H_2 \cup H_3 = G \cup Au$, where $Au$ is the second coset of $A$ in $M_1$. Pick $M_i, M_j$, with $2 \leq i < j \leq 7$. We want to show that $M_i \cap M_j \leq M_1$. Thanks to the first part of this Lemma, we know that $M_i \cap M_j \cong E_4$ or $E_8$. When $M_i \cap M_j \cong E_4$ then, trivially, $M_i \cap M_j = G \leq M_1$. When $M_i \cap M_j \cong E_8$ then $M_i \cap M_j = H_k$ for some $k \in \{1, 2, 3\}$, else $\mathcal{H}_C(G|E_8) \geq 4$, a contradiction.

Consequently, $\bigcup_{i=1}^7 M_i$ contains at least $15 + 6 \cdot 8 = 63$ involutions; 15 in $M_1$, and additional 8 in each $M_i, i > 1$. In particular, $\langle G, x \rangle \neq C$ for every involution $x$. We get
\[ \Gamma_2(E_4^+, C) = 0, \quad \Gamma_2(E_4^+, M(A_4)) = 60 - \Gamma_2(E_4^+, E_8) = 48. \]

This finishes the proof. \qed

All constants $\Gamma_i(A, B)$ have now been calculated. They are collected in Figure 1.
2.1. Random generators of arbitrary orders. According to Figure 1, there are only five orbit-nonequivalent ways to get from $e$ to $C$ in 3 steps. Namely,

- $A_0 = \{e, C_2, A_4, C\}$,
- $A_1 = \{e, C_2, E_4^-, C\}$,
- $A_2 = \{e, C_2, S_3, C\}$,
These sequences and the related constants $i(A, B)$ are visualized in Figure 2. Full lines correspond to involutions ($i = 2$), dotted lines to elements of order 3 ($i = 3$).

**Proposition 2.3.** Let $C = M^*(2)$. Then the probability that 3 randomly chosen elements of $C$ generate $C$ is $p_3(C) = 955,584 · 120^{-3} = 0.553$.

**Proof.** By (4),

$$|\text{Gen}_3(C)| = \sum_{i=0}^{4} |A_{i\sim}|.$$

By our previous calculation summarized in Figure 2, $|A_{0\sim}| = 63 · 24 · (48 + 48)$, $|A_{1\sim}| = 63 · 24 · (8 + 24)$, $|A_{2\sim}| = 63 · (32 + 32) · (18 + 36)$, $|A_{3\sim}| = 56 · 36 · (18 + 36)$, and $|A_{4\sim}| = 56 · (54 + 27) · (48 + 48)$. Thus $|\text{Gen}_3(C)| = 955,584$. We are done by (1). \hfill \Box

**2.2. Random generators of given orders.** The only possible types of orders for three generators in $C$ are $(2, 2, 2)$, $(2, 2, 3)$, $(2, 3, 3)$, and $(3, 3, 3)$. The sequences of subloops corresponding to each of these types are depicted in Figure 3. We must be careful, though, since not all combinations of lines in Figure 3 correspond to
sequences with correct types of orders. The possible continuations are emphasized in Figure 3.

**Proposition 2.4.** Let \( s = (s_1, s_2, s_3) \) be a 3-tuple of integers, \( s_1 \leq s_2 \leq s_3 \), \( C = M^*(2) \), and let \( p_s(C) \) be the probability that 3 randomly chosen elements \( a_1, a_2, a_3 \) of \( C \) generate \( C \) and \((|a_1|, |a_2|, |a_3|)\) is of type \( s \). Then \( p_{(2,2,2)}(C) = 48,384 \cdot 120^{-3} = 0.028 \), \( p_{(2,2,3)}(C) = 326,592 \cdot 120^{-3} = 0.189 \), \( p_{(2,3,3)}(C) = 435,456 \cdot 120^{-3} = 0.252 \), and \( p_{(3,3,3)}(C) = 145,152 \cdot 120^{-3} = 0.084 \).

**Proof.** Use Figure 3 and (7).

**References**


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