Calculus II Section 1: Dr. Judd
Lab 3: Numerical Integration
25 points, due Wednesday 20 February in class.

The assignment is due at the start of class Wednesday 20 February. It is your responsibility to
download the lab and work on it in a timely manner.

The only way to submit the lab is by printing it out and bringing it to class. I will not accept the
electronic file in an email, on a floppy disk, or any other way. You may work with your classmates
on this assignment, but you should produce your own document on your own computer. Taking
someone’s document and changing the name and a word or two violates our integrity policy.

The goal of this lab is to use Scientific Notebook to approximate definite integrals numerically.
The techniques that we shall use have similar ideas, and they each give a formula for the maximum
error in the approximation. They are the trapezium rule and Simpson’s rule. For this lab you will
need to set the maximum number of decimal places to at least 6 in Scientific Notebook.

Part I

Each technique starts the same way. We have a definite integral \( \int_a^b f(x) \, dx \) that we want to evaluate.
We first decide into how many sub-intervals we want to split the interval \([a, b]\)—say \(n\). Each sub-
interval will have the same length, \( h = (b - a)/n \). The length \( h \) is called the step size. This
partitions the interval \([a, b]\) into \(P = \{x_0, x_1, \ldots, x_n\}\) where
\( x_0 = a, x_1 = a + h, x_2 = a + 2h, \) and in
general \( x_k = a + kh \). Next we calculate the values of our function at the points \( x_k \). Set
\( y_0 = f(x_0), y_1 = f(x_1) \) and in general \( y_k = f(x_k) \).

We want to evaluate
\[
\int_0^3 \ln(1 + x) \, dx
\]
using six sub-intervals.

1.1 What will be the step size, \( h \)?

1.2 Write down the values \( x_0, x_1, x_2, x_3, x_4, x_5, x_6 \).

1.3 Evaluate the integrand at each point \( x_0, x_1, x_2, x_3, x_4, x_5, x_6 \) to find \( y_0, y_1, y_2, y_3, y_4, y_5, y_6 \) and
make a table with the \( x \)-values along the top row and the \( y \)-values along the bottom row.

Part II: The trapezium rule

The formula for the trapezium rule is
\[
S_T = \frac{h}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n),
\]
where \( S_T \) is the approximation to the integral. (See page 396 in the textbook for the derivation of
this formula.) We also have an expression for the error. Set the error to be
\[
E_T = \int_a^b f(x) \, dx - S_T,
\]
and suppose $|f''(x)| \leq M$ for every $x$ in $[a, b]$, then

$$|E_T| \leq \frac{(b-a)}{12} h^2 M .$$

2.1 Calculate $S_T$ for the values you found in Part I.

2.2 Find the second derivative of $f(x) = \ln(1 + x)$.

2.3 Find the maximum value, $M$, of $f''(x)$ on $[0, 3]$. This is the absolute maximum of the function $f''$ on $[0, 3]$.

2.4 We have $a = 0$ and $b = 3$. You calculated the step size $h$ in 1.1, and you calculated $M$ in 2.3.

Put these into the formula for the maximum value of the error, $|E_T|$, to find an upper bound for $|E_T|$.

2.5 Use *Scientific Notebook* to calculate the definite integral.

2.6 What is the actual error $E_T$?

**Part III: Simpson’s rule**

The formula for Simpson’s rule is

$$S_S = \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) ,$$

where $S_S$ is the approximation to the integral. (See page 399 in the textbook for the derivation of this formula.) We also have an expression for the error. Set the error to be

$$E_S = \int_a^b f(x) \, dx - S_S ,$$

and suppose $|f^{(4)}(x)| \leq M$ for every $x$ in $[a, b]$, then

$$|E_S| \leq \frac{(b-a)}{180} h^4 M .$$

3.1 Calculate $S_S$ for the values you found in Part I.

3.2 Find the fourth derivative of $f(x) = \ln(1 + x)$.

3.3 Find the maximum value, $M$, of $f^{(4)}(x)$ on $[0, 3]$. This is the absolute maximum of the function $f^{(4)}$ on $[0, 3]$.

3.4 We have $a = 0$ and $b = 3$. You calculated the step size $h$ in 1.1, and you calculated $M$ in 3.3.

Put these into the formula for the maximum value of the error, $|E_S|$, to find an upper bound for $|E_S|$.

3.5 Use *Scientific Notebook* to calculate the definite integral.

3.6 What is the actual error $E_S$?
Part IV: Approximating integrals with a given error

The formulæ for the errors of these two rules allow us to calculate how many intervals are needed in order to find the integral to within a given degree of accuracy. Suppose you want to use the trapezium rule to find \( \int_0^3 \ln(1 + x) \, dx \), accurate to three decimal places. This means that the error must be less than 0.0005, i.e. \(|E_T| < 0.0005\). From the formula, this means that

\[
\frac{(b - a)}{12} h^2 M < 0.0005 ,
\]

and re-arranging gives

\[
h < \sqrt{\frac{0.006}{(b - a)M}} .
\]

4.1 You know \( M, a, b \) from above. Use these to get an upper bound for the step size, \( h \).

4.2 We have \( h = (b - a)/n \). Using your answer to 4.1 find a lower bound for the number of intervals, \( n \).

4.3 Use the trapezium rule and the number of intervals you found in 4.2 to find \( \int_0^3 \ln(1 + x) \, dx \) to within three decimal places.

4.4 Using the formula for the error in Simpson’s rule, and an argument similar to that above, find an upper bound for the step size \( h \) in Simpson’s rule, and hence a lower bound for the number of intervals \( n \).

4.5 Use Simpson’s rule and 4.4 to find \( \int_0^3 \ln(1 + x) \, dx \) to within three decimal places.

4.6 How does this compare with the exact value?

4.7 Find \( \int_0^4 e^{-x^2} \, dx \) to within three decimal places.