The Newton—Raphson Method for finding roots

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1 Introduction

The Newton—Raphson method is an algorithm to find numerical approximations to roots of equations. For example, suppose you want to find the roots of \( f(x) = x^2 - 2 \), i.e. \( x = \pm \sqrt{2} \). The method starts with guessing a value of \( x \) close to the root, say \( x_0 = 2 \). We then have an algorithm for finding a sequence of points \( x_1, x_2, x_3, \ldots \), each point (hopefully) closer to the root than the previous.

1.1 How it works

![Figure 1: The Newton—Raphson method in action](image)

Figure 1: The Newton—Raphson method in action
Given the initial guess $x_0$ one may find the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ (see Figure 1). The central idea to the algorithm is that the $x$-intercept of the tangent line is often closer to the root than the previous point. This, then, will be the next point.

2 The Algorithm

We now give a detailed construction of the algorithm for the Newton—Raphson method. The result we show is that given a sequence of points in the method, $x_0, x_1, x_2, \ldots, x_n$, starting from the initial guess $x_0$, then next point $x_{n+1}$ will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2.1 Construction

Given a differentiable function $f$ we want to find a value of $x$ such that $f(x) = 0$. The first step is to guess a value $x_0$ close to $x$. Next calculate the slope of the tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$—this is just $f'(x_0)$, the derivative of $f$ at $x_0$. Using the point—slope formula for the equation of a line we find (see Figure 2) that the tangent line has equation

$$y - f(x_0) = f'(x_0)(x - x_0).$$

We now want to find the $x$-intercept of this line, i.e. the value of $x$ that gives $y = 0$. We set $y = 0$ and solve for $x$:

$$-f(x_0) = f'(x_0)(x - x_0)$$
$$-\frac{f(x_0)}{f'(x_0)} = x - x_0$$
$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Thus the next approximation to the root of $f$ that we seek will be

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

2.2 Iteration

There is no reason to stop after just one approximation. If we have found points $x_1, x_2, \ldots, x_n$ using this method, then to find the next point $x_{n+1}$ we perform the same calculation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Now continue in this way until the sequence $(x_n)$ converges (i.e. the numbers stop changing).
3 Example

We give an example of using the Newton—Raphson method to find $\sqrt{2}$.

3.1 Method

The function we shall use is $f(x) = x^2 - 2$. The function has roots $\pm \sqrt{2}$. We seek the positive square root, so our initial guess will be $x_0 = 2$. The algorithm tells us that given $x_n$, the next point in the sequence will be $x_{n+1} = x_n - f(x_n)/f'(x_n)$ so we must find the derivative. Here, $f'(x) = 2x$ and hence the formula becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

Table 1 shows the first four iterations. We stop after four because $\sqrt{2} = 1.414213562$ to 9 decimal places. Figure 3 is the graph of $y = f(x)$ close to $x = \sqrt{2}$ showing the tangent lines.

3.2 Using Scientific Notebook for the method

In order to use Scientific Notebook for the calculations you must define the function whose roots you want to find and write out the formula for the next point. Using the above example, type

$$f(x) = x^2 - 2$$
Point | Value | \((x_n^2 + 2)/2x_n\)
---|---|---
\(x_0\) | 2 | 1.5
\(x_1\) | 1.5 | 1.416666667
\(x_2\) | 1.416666667 | 1.414215686
\(x_3\) | 1.414215686 | 1.414213562
\(x_4\) | 1.414213562 | 1.414213562

Table 1: The first four iterations of the Newton—Raphson method

Figure 3: Tangent lines of \(y = x^2 - 2\) in the Newton—Raphson method
and select Compute/Define/New Definition. Then type
\[ f'(x) \]
and select Compute/Evaluate. Finally type
\[ n(x) = x - f(x)/f'(x) \]
and select Compute/Define/New Definition. Now you can start the process. Type
\[ n(2) \]
and select Compute/Evaluate. The result should be 1.5. Next try
\[ n(1.5) \]
and select Compute/Evaluate to get 1.416666667, or alternatively
\[ n(n(2)) \]
will give the same answer. Continue until the sequence stops changing.

Of course, since you are using a perfectly good computer algebra system you could simply enter
\[ f(x) = 0 \]
and select Compute/Solve/Numeric to get the same answer.

4 Problems

Unfortunately this method doesn’t always work, although a better choice of \( x_0 \) will often help you get out of the worst problems.

4.1 Poor choices of \( x_0 \)

Consider the function \( f(x) = 2x^2 - 12x + 17 \), whose graph is below. You want to find the smaller root.

If your first guess is \( x_0 = 3 \), then \( f'(3) = 0 \) and you cannot calculate \( x_1 \) because \( f'(3) \) is in the denominator of the formula. If your next guess is \( x_0 = 3.1 \), then you will get the larger root. Of course, from the graph it is clear that \( x_0 = 2 \) or \( x_0 = 2.5 \) would be better guesses.

4.2 A harder function

A more realistic example is \( f(x) = 2x^4 - 24x^3 + 104x^2 - 96x + 127 \). This has four roots \( r_1, r_2, r_3, r_4 \) from smallest to largest.
Figure 4: Graph of $y = 2x^2 - 12x + 17$

Figure 5: Graph of $y = 2x^4 - 24x^3 + 104x^2 - 96x + 127$
Suppose you want the third root $r_3$, which lies between 3 and 4 (in fact $r_3 = 3.541196073$). Here are some of the possible choices of $x_0$ and which root the point converges to.

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.05</td>
<td>$r_3$</td>
</tr>
<tr>
<td>2.065</td>
<td>$r_4$</td>
</tr>
<tr>
<td>2.9</td>
<td>$r_1$</td>
</tr>
<tr>
<td>3.1</td>
<td>$r_4$</td>
</tr>
<tr>
<td>3.93</td>
<td>$r_1$</td>
</tr>
</tbody>
</table>

### 4.3 For some functions there is no hope

Consider the example from the book on page 321 of

$$f(x) = \begin{cases} \sqrt{x-r}, & (x \geq r) \\ -\sqrt{r-x}, & (x < r) \end{cases}$$

where $r$ is some real number. The graph of $y = f(x)$ is shown in Figure 6 for $r = \sqrt{3}$.

![Figure 6: The graph of $y = f(x)$ for $r = \sqrt{3}$](image)

If you follow the algorithm for \textit{any} initial value $x_0 \neq r$, then

$$x_1 = 2r - x_0 ,$$

$$x_2 = 2r - x_1 = x_0 ,$$

$$x_3 = 2r - x_2 = 2r - x_0 = x_1 \quad \text{and so on.}$$
4.4 Does this mean the method is useless?

The short answer is “no”. In fact if \( y = f(x) \) is convex towards the x-axis (bulges) then convergence is guaranteed, and pretty quickly. What you must be careful to do is check that the sequence of points is converging, and that it converges to the root you seek.

5 Further Study

We saw in the previous section that the first guess \( x_0 \) must be chosen carefully. A natural question to ask is, “For a given function \( f \) which values of \( x_0 \) will lead to each root of the function?” Looking at example 4.1 it is easy to see that \( x_0 < 3 \) will return the lower root, \( x_0 > 3 \) will return the upper root and \( x_0 = 3 \) cannot be used because the derivative of the function is zero.

Other functions are more complicated. In example 4.2 we started to see that a small variation in the choice of \( x_0 \) could quickly change the root obtained. In Figure 7 is a diagram of which root is returned by the method for each choice of \( x_0 \) between 1 and 5. Points coloured red, green, blue and magenta return roots \( r_1, r_2, r_3 \) and \( r_4 \) respectively. The darker the shading, the longer the algorithm takes to converge; points shaded black do not approach within 0.05 of a root within 64 iterations.

![Figure 7: The resulting root of \( f(x) = 2x^4 - 24x^3 + 104x^2 - 96x + 127 \) for each choice of \( x_0 \)](image)

It is not surprising that when \( x_0 < 2 \) we get \( r_1 \), the smallest root. However (although you can only see some of this on the picture) between 2 and 2.2 there are infinitely many disjoint intervals where the choice of \( x_0 \) will return any of \( r_1, r_3 \) or \( r_4 \). Similarly between 3.8 and 4 there are infinitely many disjoint intervals where the choice of \( x_0 \) will return any of \( r_1, r_2 \) or \( r_4 \).

What is impossible to see from Figure 7 is the way the small intervals repeat themselves. If you are not familiar with complex numbers, then think of the rest of this section as a two-dimensional version of the same work. The function \( f(z) = z^6 - 1 \) defined over the complex numbers has six roots: \( \pm 1, \pm (1/2) + (\sqrt{3}/2)i \) and \( \pm (1/2) + (\sqrt{3}/2)i \), where \( i = \sqrt{-1} \). Figure 8 shows the same mapping as above for the roots of \( f \) in the complex plane. This is the example on page 323 of the book, but with the correct orientation and colouring. Points coloured red go to 1, magenta points to -1, green points \( (1/2) + (\sqrt{3}/2)i \), blue points to \(-(1/2) + (\sqrt{3}/2)i \), and so on. Points which generate sequences that do not converge to within \( 10^{-16} \) units of a root after 64 iterations are coloured black.
Figure 8: The resulting root of $f(z) = z^6 - 1$ for each choice of $z_0$
The graph is the region \( \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\} \) in the \( xy \)-plane. Notice how there are certain places where the root you end up with can change rapidly over a very small area. Furthermore, the same patterns are repeated throughout the region. These two properties are typical of fractals and this is indeed an example of a fractal.