4.4 Riemann Sums

Goals

- Understand “Sigma notation”
- Find Riemann sums using different numbers of rectangles
- Calculate the average value of a function

Problems

1. Write the sum \( \sum_{k=2}^{4} k^2 \) without sigma notation and evaluate it.

2. Graph the function \( f(x) = 2 + x - x^2 \) over the interval \([-1, 2]\). Partition the interval into 6 sub-intervals of equal length. Then add to your graph rectangles for the left, right and midpoint Riemann sums (draw different sketches for each sum). Evaluate each of these sums.

3. Graph the function \( f(x) = 1 + \sqrt{1 - (x - 2)^2} \) over the interval \([1, 3]\), and use areas to evaluate \( \int_{1}^{3} 1 + \sqrt{1 - (x - 2)^2} \, dx \).

4. What is the formula for the average value of \( f(x) \) over the interval \([a, b]\)?

Food for thought

We have defined the area between the \( x \)-axis and a non-negative function \( f \) over the interval \([a, b]\) to be the definite integral \( \int_{a}^{b} f(x) \, dx \). What happens if you try to use this formula for the function \( f(x) = \sin(x) \) on the interval \([0, \pi]\)? What answer do you get? Why? (Think about the Riemann sum.) How can you re-write the formula to correctly give the area enclosed between the \( x \)-axis and the curve?

The function \( f(x) = x^3 - x^2 - 2x \) on \([-1, 2]\) presents similar problems (graph it). Use the ideas from the previous paragraph to write down a formula using definite integrals for the area enclosed between the \( x \)-axis and the graph of \( y = f(x) \).

4.5 The Fundamental Theorem of Calculus

Goals

- Understand the Fundamental Theorem of Calculus
- Apply the theorem to calculate definite integrals
Problems

1. State the Fundamental Theorem of Calculus

2. Use the Fundamental Theorem of Calculus to evaluate

   \( \int_{4}^{9} \frac{1 - \sqrt{x}}{\sqrt{x}} \, dx \) , \quad \( \int_{-\pi/2}^{\pi/2} 8x^2 + \sin x \, dx \) .

3. Find \( dy/dx \) for

   \( \int_{1}^{x^{1/3}} e^{t^3+1} \, dt \) , \quad \( \int_{\sqrt{x}}^{0} \sin t^2 \, dt \) .

4.6 Substitution in Definite Integrals

Goals

- Be able to use integration by substitution correctly for definite integrals.
- Find areas between curves.

Problems

1. Use substitution to find

   \( \int_{1}^{-1} \frac{5x}{(4 - x^2)^2} \, dx \) , \quad \( \int_{0}^{\pi} e^{\sin x} \cos x \, dx \) .

2. Find the region enclosed between the curves

   (a) \( y = 7 - 2x^2 \) and \( y = x^2 \),
   
   (b) \( y = \sin(\pi x/2) \) and \( y = x \).

Food for thought

The pair of curves \( f(x) = x^4 - 4x^2 - 4 \) and \( g(x) = x^2 \) intersect four times. Find these four points. Find on which intervals \( f(x) \geq g(x) \) and on which intervals \( g(x) \geq f(x) \). Use this information to write down a formula using definite integrals to calculate the area enclosed between the two curves. Find the area.