Calculus III, Dr. Judd
Lab 2: Sequences
25 points, due Tuesday 16 April in class.

The assignment is due at the start of class Tuesday 16 April. It is your responsibility to download the lab and work on it in a timely manner.

The only way to submit the lab is by printing it out and bringing it to class. I will not accept the electronic file in an email, on a floppy disk, or any other way. You may work with your classmates on this assignment, but you should produce your own document on your own computer. Taking someone’s document and changing the name and a word or two violates our integrity policy.

The goals of this lab are to begin thinking about the qualitative behaviour of sequences; to gain familiarity with some important but non-obvious limits; and to develop a feel for various rates of growth of sequences that diverge to infinity.

For each sequence that you investigate you should plot enough points (at least 50) to get a feel for its behaviour. As an example, to plot the sequence \( a_n = (1 + 1/n)^n \) (shown below) position the cursor after the expression, and select Compute/Plot2D/Rectangular. Then select the Items Plotted tab and set the Plot Style to Point, the Point Marker to Circle, then click on the Variables and Intervals button and set the Plot Interval to 1 to 50, and the Sample Size to 50.

Unfortunately this doesn't quite work, but it gives you a good idea. For the problems involving \((-1)^n\) replace \((-1)^n\) with \(\cos(\pi n)\). (Why does this work?) Scientific Notebook will plot too many points, but you will still get the idea. Alternatively go to the help and look in the section on sequences and series for how to plot sequences with Maple.

Introduction

Let \( (a_n) \) be a sequence of real numbers. For the purpose of this lab we distinguish four cases. If \( a_n \) increases to \( \infty \) as \( n \to \infty \), we say \( (a_n) \) diverges to infinity; if \( a_n \) decreases to \( -\infty \) as \( n \to \infty \), then we say \( (a_n) \) diverges to negative infinity; if \( \lim_{n \to \infty} a_n = c \), a finite real number, we say that \( (a_n) \) converges to \( c \); and in all other cases, we say that \( (a_n) \) diverges by oscillation.
1. Before you begin the lab, decide which of the four types of behaviour applies to each of the following sequences.

(a) \[ a_n = \frac{n}{\sqrt{n}} \]

(b) \[ a_n = \frac{n + (-1)^n}{n} \]

(c) \[ a_n = \frac{2^n + n^3}{3^n + n^2} \]

(d) \[ a_n = \frac{(-1)^n(n - 1)}{n} \]

2. Use the computer to plot the first 50 terms of each of the four sequences in Problem 1 as described above. You may need to adjust the vertical scale so that what you see is consistent with what you expected; or perhaps you might need to rethink your answers to Problem 1. Use the same graphical analysis to investigate the behaviour of each of the following sequences

(a) \[ a_n = \frac{\ln n}{\sqrt{n}} \]

(b) \[ a_n = \frac{n^{10}}{2^n} \]

(c) Experiment with several other values of \( k \) for sequences of the form \( a_n = n^k/2^n \). Does changing the value of \( k \) seem to affect the value of the limit?

3. In this problem we investigate the behaviour of the sequence \( (r^n) \) where \( r \) is any real number.

(a) Write out the first few terms of the sequence \( (r^n) \) for \( r = 1 \) and \( r = -1 \). (No need for the machine on this one.) Which one does not converge?

(b) Determine the behaviour of \( (r^n) \) for other values of \( r \), both positive and negative, keeping track of the behaviour you observe: divergent to infinity or negative infinity, convergent to a finite real number (which?), or divergent by oscillation.

(c) Tell the whole story. In your written report for this lab, give a complete description of all possible types of behaviour for the sequence \( (r^n) \) as \( r \) ranges over all real numbers. State clearly—with specific examples—which values of \( r \) give rise to which type of behaviour.

4. The race to infinity: although many sequences simply “diverge to infinity” they may do so at different rates. In this problem we clarify the notion of “relative rates”.

Let \( (a_n), (b_n) \) be sequences of positive real numbers that diverge to infinity. We say \( (a_n) \) is strictly faster than \( (b_n) \) if and only if the sequence \( (a_n/b_n) \) diverges to infinity, and \( (a_n) \) is strictly slower than \( (b_n) \) if and only if \( \lim_{n \to \infty} a_n/b_n = 0 \). If \( \lim_{n \to \infty} a_n/b_n \) is finite and non-zero, then we say that \( (a_n) \) is comparable to \( (b_n) \).

On the basis of your earlier work in this lab, which is strictly faster, \( (\sqrt{n}) \) or \( (\ln n) \)? Are \( (2^n) \) and \( (n^2) \) comparable? What about \( (2n - 3) \) and \( (3n - 2) \)? Explain fully in complete sentences.

Robert Judd, 5 April 2002.