Calculus III, Dr. Judd
Lab 3: Introduction to Series
25 points, due Tuesday 30 April in class.

The assignment is due at the start of class Tuesday 30 April. It is your responsibility to download the lab and work on it in a timely manner.

The only way to submit the lab is by printing it out and bringing it to class. I will not accept the electronic file in an email, on a floppy disk, or any other way. You may work with your classmates on this assignment, but you should produce your own document on your own computer. Taking someone’s document and changing the name and a word or two violates our integrity policy.

Goals

• To understand series convergence in terms of limits of partial sums.
• To explore the family of $p$-series.
• To investigate the growth of the partial sums for the harmonic series.

In the Lab In trying to attach meaning to an infinite series $\sum_{k=1}^{\infty} b_k$ of real numbers $b_k$, it is natural to study the behavior of the partial sums $S_N = \sum_{k=1}^{N} b_k$. If $\lim_{n \to \infty} S_N$ exists, we say that the series converges and we define the value of the limit to be the sum of the infinite series. Otherwise, we say the series diverges.

1. Consider the geometric series $\sum_{k=0}^{\infty} r^k$, where the real number $r$ gives the ratio between successive terms. Note that series starts with $k = 0$ rather than $k = 1$.

   (a) Let $r = \frac{4}{5}$ and use the computer to investigate the partial sums of the series $\sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k$. As you evaluate partial sums with more and more terms, what appears to be the limiting value? Confirm that this value for the sum of the infinite series agrees with the well-known formula for the sum of a geometric series whose ratio $r$ has an absolute value less than 1.

   (b) Let $r = 1.01$ and use the computer to study the partial sums. What do you conclude about the convergence or divergence of the geometric series in this case? Explain.

   (c) Now let $r = 1$ and $r = -1$. Discuss the convergence or divergence of the series in these cases. Do you really need to use a computer in part c or even in part b? Explain.
(d) When $r$ is negative in a geometric series we get an example of an \textit{alternating series}. Explain why this is an aptly chosen description. Give examples of two alternating geometric series; one convergent and one divergent.

2. The \textit{p-series}. Now consider the \textit{p-series} $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where $p$ can be any positive real number.

   (a) For $p = 0.5$ and $p = 2$, write down the first five terms of each sequence. Then, by studying partial sums, argue that one of these series converges and one diverges. Do not get too ambitious here on the number of terms, $N$, in your partial sums; SN may refuse the calculate the number or take a long time. Give an estimate of the sum for the convergent series. [Obscure hint indicating a beautiful and mysterious fact: multiply the partial sums by 6 and take the square root of the results.]

   (b) Select two more $p$ values, one of which gives a convergent $p$-series and the other a divergent one. Give explanations and evidence for your conclusion.

   (c) When $p = 1$, we get a special and important case of the \textit{p-series} known as the \textit{harmonic series} $\sum_{k=1}^{\infty} \frac{1}{k}$. Write out and then compute the first four terms of the harmonic series. Then, using SN, estimate the partial sums for $N = 4$, $N = 128$, and $N = 1024$.

   (d) Based on your results of part a and b, formulate a preliminary conjecture about convergence and divergence of the $p$-series in terms of the positive real number $p$. Ignore the case $p = 1$ for now. That is the subject of the next problem.

3. The \textit{harmonic series}. It is a remarkable and important fact that the harmonic series diverges. Where you persuaded otherwise by the fact that its terms tend to 0? Any straightforward attempt to compute this sum by a computer, no matter how large or powerful, will lead to the incorrect conclusion that the series converges. In this problem we will use the computer to obtain a valuable hint about how to prove that the harmonic series diverges.

   (a) Building on what you did in Problem 2c, compute the partial sums of the harmonic series with $N = 4, 8, 16, 32, 64$ and 128. Confirm that each time enough successive terms are added for $N$ (the number of terms in the partial sum) to reach the next power of two, the partial sum increases by an amount exceeding $\frac{1}{2}$.

   (b) Assuming that this subtle but systematic process of increasing by $\frac{1}{2}$ persists for arbitrarily large $N$, argue, with no help from the computer, that the values of the partial sums will eventually exceed 14.
Give a value of $N$ for which the partial sum exceeds 14. Justify your answer.

(c) Try to prove what you observed in part a: that by successively adding enough terms to any partial sum, we can further increase its value by at least $\frac{1}{2}$. Hint: consider

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

and

$$\frac{1}{5} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}.$$ 

How many terms are in the first sum and what is the smallest term? What about the second sum? Finally, think similarly about the general case,

$$\frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \ldots + \frac{1}{2^{k+1}}$$

(d) Now, refine your answer to Problem 2d with a more definitive statement about the convergence and divergence of the $p$-series.