1 CALCUULUS III, Spring 2002

LIST OF USEFUL TAYLOR SERIES

\begin{itemize}
\item $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$, for $|x| < 1$.
\item $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x^2 + x^4 - x^6 + \cdots$, for $|x| < 1$.
\item $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, for all $x$.
\item $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, for all $x$.
\item $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$, for all $x$.
\item $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$, for $|x| < 1$.
\item $\tan^{-1}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $|x| < 1$.
\end{itemize}

(I) Use the list of “useful Taylor series” and simple substitution to find the Taylor series of the following functions. Then use the formula $a_n = \frac{f^{(n)}(0)}{n!}$ to find the sixth derivative at $x = 0$ of each function:

1. $\ln(1-x)$ in powers of $x$.
2. $\sin(3x)$ in powers of $x$.
3. $\frac{1}{1+x^2}$ in powers of $x$.
4. $e^{-2x}$ in powers of $x$.
5. $\tan^{-1}(\frac{x}{2})$ in powers of $x$.

(II) Use the list of “useful Taylor series” to find the Taylor series of the following functions, and, if possible, find the 44th derivative at $x = c$ ($c$ is $-2$ for the first problem, $-1$ for the second, and 0 for the rest).

1. $\frac{1}{2}$ in powers of $(x+2)$.
2. $\frac{2}{x+5}$ in powers of $(x+1)$.
3. $x^3 e^{-2x}$ in powers of $x$.
4. $e^{2x} + e^{-x}$ in powers of $x$.
5. $\sin x$ in powers of $x$.
6. $e^{x} - (1 + x)$ in powers of $x$.
7. $(1 + x) \ln (1 + x)$ in powers of $x$.
8. $\ln \left( \frac{1+x}{1-x} \right)$ in powers of $x$. (Hint: $\ln \left( \frac{1+x}{1-x} \right) = \ln (1 + x) - \ln (1 - x)$).
9. $\frac{5+2x}{x^2+2}$ in powers of $x$. (Hint: divide the polynomials first).
10. $\frac{5+x}{2-x-x^2}$ in powers of $x$. (Hint: use partial fractions).
11. $\ln [(1 + 2x)(1 + 3x)]$ in powers of $x$.
12. $\cos^2(x)$ in powers of $x$. (Hint: use the double angle formula).
(III) Integrate or differentiate a known power series to find the Taylor series at \( x = 0 \) for the following:

1. \( F(x) = \int_0^x \frac{1}{1+x} \, dx \) (Does it look familiar?)
2. \( F(x) = \int_0^x e^{x^2} \, dx \)
3. \( \frac{d}{dx} (x^3 \tan^{-1}(x)) \)
4. \( \frac{1}{(1-x)^2} \) (Hint take the second derivative of \( \frac{1}{1-x} \))

(IV) Find the first three terms of the Taylor series (in powers of \( x \)) of the following series (i.e., find the cubic approximation).

1. \( (\sin x) (\cos x) \)
2. \( \frac{x^2-6x+7}{(1-x)(2-x)(3-x)} \)
3. \( (\sin x) (\tan^{-1}(2x)) \)
4. \( \frac{\sin x}{\cos x} \)

(V) Find the first terms of the power series of the following functions to compute the limit

1. \( \lim_{x \to 0} \frac{e^{x/(1+x)} - 1}{x} \)
2. \( \lim_{x \to 0} \frac{1 - \cos(2x) - (x^2/2)}{x^4} \)
3. \( \lim_{x \to 0} \frac{\tan^{-1}(x) - \sin(x)}{x^3 \cos(x)} \)

(VI) Write a power series of \( f(x) = e^x - 1 \) in powers of \( x \). Then differentiate the power series and show that

\[
\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1
\]

(VII) Write a power series of \( f(x) = xe^x \) in powers of \( x \). Then integrate the power series and show that

\[
\sum_{n=1}^{\infty} \frac{1}{n!(n+2)} = \frac{1}{2}
\]

(VIII) Suppose that the function \( f \) is infinitely differentiable on an interval containing \( x = 0 \). Suppose that \( f(0) = 1 \) and that \( f'(x) = 2f(x) \) for all \( x \). Use these properties to find the following:

1. \( f'(0) \), \( f''(0) \), and \( f'''(0) \)
2. \( f^{(n)}(0) \) (a formula for the \( n \)th derivative of \( f \) at zero)
3. The series representation of \( f(x) \) in powers of \( x \).