Convergence of Power Series

A power series centred at \( x = c \) is an infinite series
\[
\sum_{n=0}^{\infty} a_n (x - c)^n .
\]

For each such series there exists a number \( R, 0 \leq R \leq \infty \), called the \textit{Radius of Convergence}, such that the series converges when \( c - R < x < c + R \), and diverges when \( x < c - R \) and \( c + R < x \). The points \( x = c \pm R \) must be tested separately. The radius of convergence is calculated by applying the ratio test to the series:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}(x - c)^{n+1}}{a_n(x - c)^n} \right| = \lim_{n \to \infty} |x - c| \left| \frac{a_{n+1}}{a_n} \right| = |x - c| \cdot L \text{ where } L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|. 
\]

By the ratio test the series converges when \( |x - c| \cdot L < 1 \), i.e. \( |x - c| < 1/L \). The radius of convergence is thus \( R = 1/L \).

Differentiation and Integration of Power Series

Power series may be integrated and differentiated “term by term”, that is, each term evaluated separately.
\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n (x - c)^n \right) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} (x - c)^n .
\]

\[
\int \left( \sum_{n=0}^{\infty} a_n (x - c)^n \right) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1} + \text{constant} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (x - c)^n + \text{constant} .
\]

As an example, consider the power series for \( \sin x \) and \( \cos x \):

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} , \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} .
\]

Taylor Series

The Taylor series centred at \( x = c \) for an infinitely differentiable function \( f(x) \) is given by
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n ,
\]
where \( f^{(n)}(c) \) is the \( n \)th derivative of \( f \) at \( c \). The \( N \)th Taylor polynomial of \( f \) is the \( N \)th partial sum of the series
\[
T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots .
\]

The \( N \)th remainder is just the difference between the function and its \( N \)th Taylor polynomial
\[
R_N(x) = f(x) - T_N(x) .
\]

Thus \( f(x) \) is equal to its Taylor series on \( (c - R, c + R) \) whenever \( \lim_{n \to \infty} R_N(x) = 0 \) for all \( x, |x - c| < R \). Taylor’s inequality for the remainder says that if \( |f^{(N+1)}(x)| \leq M \) for \( |x - c| < R \), then
\[
|R_N(x)| \leq \frac{M}{(N+1)!} |x - c|^{N+1}
\]
whenever \( |x - c| < R \).

To approximate a function \( f \) at \( x = c \) to within a given error, say \( \varepsilon = 0.001 \), you must find a value of \( N \) for which \( |R_N(x)| < \varepsilon = 0.001 \) whenever \( |x - c| < R \).