Sequences

**Definition.** The sequence \((x_n)_{n=1}^\infty\) converges to the limit \(L\) if for every \(\varepsilon > 0\) there exists \(N \geq 1\) such that \(|x_n - L| < \varepsilon\) whenever \(n \geq N\).

**Theorem.** If \(x_n \to L\) and \(y_n \to M\) as \(n \to \infty\), and \(a, b \in \mathbb{R}\), then
- \(ax_n + by_n \to aL + bM\) as \(n \to \infty\),
- \(x_n y_n \to LM\) as \(n \to \infty\),
- \(x_n / y_n \to L/M\) as \(n \to \infty\) provided \(M \neq 0\).

**Theorem (Sandwich Theorem).** If both sequences, \((x_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) converge to \(L\), and \(x_n \leq z_n \leq y_n\) for all \(n\), then \((z_n)_{n=1}^\infty\) converges to \(L\) also.

**Definition.** The sequence \((x_n)_{n=1}^\infty\) is:
- increasing if \(x_n \leq x_{n+1}\) for all \(n\).
- decreasing if \(x_n \geq x_{n+1}\) for all \(n\).
- strictly increasing if \(x_n < x_{n+1}\) for all \(n\).
- strictly decreasing if \(x_n > x_{n+1}\) for all \(n\).

A sequence which is either increasing or decreasing is called monotone.

**Theorem (Monotone convergence theorem).**
- If \((x_n)_{n=1}^\infty\) is increasing and bounded above, then it converges.
- If \((x_n)_{n=1}^\infty\) is decreasing and bounded below, then it converges.

**Properties of convergent sequences**
- A sequence can have at most one limit.
- If \(x_n \to L\) as \(n \to \infty\), and \(a \leq x_n \leq b\) for every \(n\), then \(a \leq L \leq b\).
- A convergent sequence is bounded.

**Examples.**
- If \(r > 0\), then \(1/n^r \to 0\) as \(n \to \infty\).
- If \(|a| < 1\), then \(a^n \to 0\) as \(n \to \infty\).
- If \(x_n \to \pm \infty\) as \(n \to \infty\), then \(1/x_n \to 0\) as \(n \to \infty\).

Series

**Definition.** The partial sums of the series \(\sum_{n=1}^\infty a_n\) are the numbers \(S_N = \sum_{n=1}^N a_n\). The series \(\sum_{n=1}^\infty a_n\) converges with sum \(S\) if and only if the sequence of partial sums \((S_N)_{n=1}^\infty\) converges to the limit \(S\). Otherwise the series diverges.

**Theorem.** If \(\sum a_n\) converges to \(A\), and \(\sum b_n\) converges to \(B\), then \(\sum (pa_n + qb_n)\) converges to \(pA + qB\). In particular,
- \(\sum (a_n + b_n) = \sum a_n + \sum b_n\),
- \(\sum (a_n - b_n) = \sum a_n - \sum b_n\),
- \(\sum pa_n = p\sum a_n\).
Theorem. If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$. The converse is false; if $a_n \to 0$ then $\sum a_n$ need not converge. Remember that $\sum 1/n$ diverges.

Theorem. The following are equivalent:
- $\sum a_n$ converges,
- $\sum a_n$ converges for some $N \geq 1$,
- $\sum a_n$ converges for every $N \geq 1$.

Definition. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. $\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges. $\sum a_n$ diverges otherwise.

Theorem. If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Convergence tests for series

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<th>Series</th>
<th>Converges</th>
<th>Diverges</th>
<th>Comment</th>
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<tbody>
<tr>
<td>$n^{th}$-term</td>
<td>$\sum_1^\infty a_n$</td>
<td></td>
<td>$\lim_n a_n \neq 0$</td>
<td>cannot use to show convergence</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\sum_0^\infty ax^n$</td>
<td>$</td>
<td>x</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td>Telescoping</td>
<td>$\sum_1^\infty (b_n - b_{n+1})$</td>
<td>$\lim_n b_n = L$</td>
<td>no limit for $(b_n)_1^\infty$</td>
<td>sum $S = b_1 - L$</td>
</tr>
<tr>
<td>$p$-series</td>
<td>$\sum_1^\infty 1/n^p$</td>
<td>$p &gt; 1$</td>
<td>$0 &lt; p \leq 1$</td>
<td></td>
</tr>
<tr>
<td>Integral</td>
<td>$\sum_1^\infty a_n, f(n) = a_n$</td>
<td>$\int_1^\infty f(x) , dx$</td>
<td>converges</td>
<td>$\int_1^\infty f(x) , dx$</td>
</tr>
<tr>
<td>Alternating</td>
<td>$\sum_1^\infty (-1)^n a_n$</td>
<td>$0 \leq a_{n+1} \leq a_n, \lim_n a_n = 0$</td>
<td>$0 \leq a_{n+1} \leq a_n, \lim_n a_n = 0$</td>
<td></td>
</tr>
<tr>
<td>Ratio</td>
<td>$\sum_1^\infty a_n$</td>
<td>$\lim_n</td>
<td>a_{n+1}/a_n</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td>Root</td>
<td>$\sum_1^\infty a_n$</td>
<td>$\lim_n \sqrt[n]{</td>
<td>a_n</td>
<td>} &lt; 1$</td>
</tr>
<tr>
<td>Direct comparison</td>
<td>$\sum_1^\infty a_n$</td>
<td>$0 \leq a_n \leq b_n, \sum b_n$ converges</td>
<td>$0 \leq b_n \leq a_n, \sum b_n$ diverges</td>
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</tr>
<tr>
<td>Limit comparison</td>
<td>$\sum_1^\infty a_n$</td>
<td>$\lim_n a_n/b_n = L, \sum b_n$ converges</td>
<td>$\lim_n a_n/b_n = L, \sum b_n$ diverges</td>
<td></td>
</tr>
</tbody>
</table>

Some pointers for which test you should use:

1. Does the $n^{th}$ term go to zero? If not, then the series diverges.
2. Is the series a “special type”: geometric, $p$-series, telescoping, alternating?
3. Can you apply the integral, ratio or root tests?
4. Can the series be compared “favourably” with one of the “special types”?