MATH 3451 Homework Assignment 1 solutions

Section 1.2:

5. Since this function is piecewise defined, we cannot just take the derivative using Quotient/Chain rules. By definition,

$$B'(0) = \lim_{x \to 0} \frac{B(x) - B(0)}{x - 0}.$$

There are two one-sided limits here. One is simple:

$$\lim_{x \to 0^-} \frac{B(x) - B(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0.$$

The other is

$$\lim_{x \to 0^+} \frac{B(x) - B(0)}{x - 0} = \lim_{x \to 0^+} \frac{e^{-1/x^2}}{x}.$$

The top and bottom both approach 0 as $x \to 0$, so this is an indeterminate form. If we try to use L'Hospital's Rule directly, things get worse:

$$\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0} \frac{(e^{-1/x^2})'}{x'} = \lim_{x \to 0^+} \frac{2x^{-3}e^{-1/x^2}}{1} = \lim_{x \to 0} \frac{2e^{-1/x^2}}{x^3}.$$

Continuing in this way will only yield larger and larger exponents in the denominator, and will always give an inevaluable $0/0$ indeterminate form. There are several ways to resolve this, but the easiest is to rewrite the original limit using negative exponents:

$$\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0^+} \frac{x^{-1}}{e^{1/x^2}} = \lim_{x \to 0^+} \frac{(x^{-1})'}{(e^{1/x^2})'} = \lim_{x \to 0^+} \frac{-x^{-2}}{-2x^{-3}e^{1/x^2}} = \lim_{x \to 0^+} \frac{x}{2e^{1/x^2}}.$$

Now, the numerator approaches 0 and the denominator approaches ∞, so this limit is 0, meaning that the original one-sided limit $\lim_{x \to 0^+} \frac{B(x)-B(0)}{x-0}$ is also 0. Since the two one-sided limits are each equal to 0, $B'(0) = 0$.

Section 1.3:

3(a). Clearly $x = 0$ is a fixed point, and for any $x \neq 0$, $|f(x)| = \frac{|x|}{2} < |x|$. Therefore, for any $x \neq 0$, $|f^n(x)| < |f^{n-1}(x)| < \cdots < |x|$ for all $n > 0$, so $f^n(x) \neq x$ and x is not periodic. Therefore, $x = 0$ is the only periodic point.

3(b). Clearly $x = 0$ is a fixed point, and for any $x \neq 0$, $|f(x)| = 3|x| > |x|$. Therefore, for any $x \neq 0$, $|f^n(x)| > |f^{n-1}(x)| > \cdots > |x|$ for all $n > 0$, so $f^n(x) \neq x$ and x is not periodic. Therefore, $x = 0$ is the only periodic point.

3(c). Clearly $x = 0$ is a fixed point, and for any $x \in (0, 1]$, $f(x) = x - x^2 < x$, but $f(x) = x(1 - x) \geq 0$, so $f(x) \in (0, 1]$ as well. Therefore, by induction, for any $x \neq 0$, $f^n(x) < x$ for all $n > 0$, so x is not periodic. Therefore, $x = 0$ is the only periodic point.

3(f). Clearly $x = -1, 0, 1$ are all fixed points. For any $x \in (0, 1)$, $f(x) = \frac{1}{2}(x + x^3) < \frac{1}{2}(2x) = x$, and $f(x) > 0$, so $f(x) \in (0, 1)$. Therefore, by induction, for any $x \in (0, 1)$, $f^n(x) < x$ for all $n > 0$, so x is not periodic. Similarly, for any $x \in (-1, 0)$, $f(x) = \frac{1}{2}(x + x^3) > \frac{1}{2}(2x) = x$, and $f(x) < 0$, so $f(x) \in (-1, 0)$. Therefore, by induction, for any $x \in (-1, 0)$, $f^n(x) > x$ for all $n > 0$, so x is not periodic. Therefore, $x = -1, 0, 1$ are the only periodic points.

4(a). The stable set of 0 is $\mathbb{R}$; for any $x \neq 0$, $f^n(x) = (-1/2)^n x$, which approaches 0 as $n \to \infty$.

4(b). The stable set of 0 is $\{0\}$; for any $x \neq 0$, $|f^n(x)| = 3^n x$, which approaches ∞ as $n \to \infty$, and therefore $f^n(x)$ does not approach 0.

4(c). The stable set of 0 is $[0, 1]$; as shown before, $f^n(x)$ is decreasing and nonnegative for all $x \in [0, 1]$, and so must approach a limit. But that limit must then be a fixed point, and 0 is the only one. Since $f^n(x)$ approaches 0 for all $x \in [0, 1]$, $[0, 1]$ is the stable set of 0.

4(f). The stable set of 0 is $(0, 1)$, the stable set of -1 is $\{-1\}$, and the stable set of 1 is $\{1\}$. To prove this, it suffices to show that for all $x \in (-1, 1)$, $f^n(x)$ approaches 0. Recall that if $x \in (0, 1)$, then $f^n(x)$ is decreasing and positive, and therefore it approaches a limit. This limit must be a fixed point, and 0 is the only possible value. Therefore, $f^n(x) \to 0$. A similar argument shows that for $x \in (-1, 0)$, $f^n(x) \to 0$ as well. Finally, 0 is a fixed point, so trivially $f^n(0) \to 0$.

7. If f is a homeomorphism, then as discussed in class, either f is increasing on all of $\mathbb{R}$ or decreasing on all of $\mathbb{R}$. If f is increasing, then for all $x < y$, $f(x) < f(y)$. Suppose that x is not a fixed point of f. Then, either $f(x) > x$ or $f(x) < x$. We treat only the first case, as the second is trivially similar. Since $f(x) > x$ and f is increasing, $f(f(x)) = f^2(x) > f(x)$. By induction, the orbit $f^n(x)$ is increasing. But then clearly $f^n(x) > x$ for all n, so x is not periodic. We have shown that the only possible periodic points for increasing f are fixed points.

Now, suppose that f is decreasing, i.e. for all $x < y$, $f(x) > f(y)$. Then clearly f^2 is increasing; if $x < y$, then $f(x) > f(y)$, implying that $f^2(x) < f^2(y)$. Suppose that f has a periodic point x, i.e. $f^n(x) = x$. Then clearly $(f^2)^n(x) = f^{2n}(x) = (f^n)^2(x) = x$, so x is a periodic point of f^2 as well. But since f^2 is increasing, this means that x is a fixed point of f^2, so $f^2(x) = x$ and x has least period either 1 or 2. We've shown that the only possible periodic points for decreasing f have least period 1 or 2.

There are many examples we could use for a homeomorphism with points of least period 2. For instance, $f(x) = -x$ is a homeomorphism (it's its own inverse), but every nonzero x has least period 2 for f; $f^2(x) = -(f(x)) =$

$-(-x) = x$.

11. We claim that every number of the form $x = \frac{i}{2^n}$ for positive integers i, n satisfying $0 \leq i < 2^n$ is eventually fixed for f. To see this, simply note that $f^n(x) = 2^n x \pmod 1 = \frac{2^n i}{2^n} \pmod 1 = i \pmod 1 = 0$, and that 0 is fixed for f.

To see that the set $S = \{\frac{i}{2^n} \ : \ 0 \leq i < 2^n\}$ is dense, choose any interval $(a, b) \subset [0, 1)$. Since $2^{-n} \to 0$, there exists N so that $2^{-N} < b - a$. For this fixed N, the numbers $0, \frac{1}{2^N}, \frac{2}{2^N}, \ldots, \frac{2^N-1}{2^N}$ have gaps 2^{-N} between closest values, and so since the length of (a, b) is more than 2^{-N}, there exists $0 \leq i < 2^N$ so that $\frac{i}{2^N} \in (a, b)$. Since (a, b) was arbitrary, this shows that S is dense.