MATH 3451 Homework Assignment 2 solutions

Section 1.4:

1(a). As shown in problem 1.3.4(c) from last week, the only periodic point for $f(x) = x - x^2$ is $x = 0$. At $x = 0$, $|f'(x)| = |1 - 2x| = 1$, so we can't conclude whether $x = 0$ is attracting or repelling with only the derivative. However, recall that it was also shown in problem 1.3.4(c) that for all $x \in [0, 1]$, $f^n(x) \to 0$ as $n \to \infty$, so $x = 0$ is certainly 'attracting from the right-hand side'. If $x < 0$, then $f(x) = x - x^2 < x < 0$, so by induction $f^n(x)$ is decreasing. If $f^n(x)$ were bounded from below, it would have to approach a fixed point, which is impossible since all are less than $x < 0$ and 0 is the only fixed point. Therefore, $f^n(x) \to -\infty$ in this case. So, $x = 0$ is 'repelling from the left-hand side', and it's then easy to see that it's neither attracting nor repelling.

1(b). The results of this section show that $f(x) = 2(x - x^2)$ has two fixed points, at $x = 0$ and $x = p_2 = 1 - \frac{1}{2} = \frac{1}{2}$. We show below in problem 1.5.1 that for all $x \in (0, 1)$, $f^n(x) \to \frac{1}{2}$, so no point in $(0, 1)$ is periodic except $\frac{1}{2}$. $x = 1$ is not periodic since $f(x) = 0$, a fixed point. Finally, since $f(x) = \mu x(1 - x)$ for $\mu > 1$, all $x \notin [0, 1]$ have $f^n(x) \to -\infty$ by a result from section 1.5 (also proved in class), so none of those are periodic. We've shown that $x = 0$ and $x = \frac{1}{2}$ are the only periodic points. At $x = 0$, $|f'(x)| = |2(1 - 2x)| = 2 > 1$, so $x = 0$ is repelling. Similarly, at $x = \frac{1}{2}$, $f'(x) = |2(1-2x)| = 0 < 1$, so $x = \frac{1}{2}$ is attracting.

1(g). First, note that if we define $g(x) = e^{x-1} - x$, then $g'(x) = e^{x-1} - 1$, which is clearly negative for $x < 1$, zero at $x = 1$, and positive for $x > 1$. Therefore, g has a global minimum at $x = 1$. Also note that $g(1) = 0$. Therefore, g is nonnegative for all x, and $g(1) = 0$ is the only root. This means that if $f(x) = e^{x-1}$, then $f(x) \geq x$ for all x, and $x = 1$ is the only place where $f(x) = x$, i.e. the only fixed point. But this means that for every x, $f^n(x)$ is nondecreasing, and that $x = 1$ is the only periodic point.

At $x = 1$, $|f'(x)| = e^{x-1} = 1$, so we can't conclude whether $x = 0$ is attracting or repelling with only the derivative. However, similarly to (a), if $x < 1$, then $f(x) > x$, but $f(x) = e^{x-1} < e^0 = 1$, so by induction $f^n(x)$ is increasing and bounded from above by 1. It therefore approaches a limit, which must be a fixed point, which must be $x = 1$. So, $x = 1$ is 'attracting from the left-hand side.' If $x > 1$, then $f^n(x)$ is increasing, and if it were bounded from above, it would approach a fixed point, which is impossible since $x = 1$ is the only fixed point. Therefore, in this case $f^n(x) \to \infty$, so $x = 1$ is 'repelling from the right-hand side.' So, overall $x = 1$ is neither attracting nor repelling.

Section 1.5:

1. It is useful to note for the remainder of this problem that $x = 0$ and $x = \frac{1}{2}$ are fixed points of $f(x) = 2x(1 - x)$, and that $f(\frac{1}{2}) = \frac{1}{2}$ is the global maximum of f. If $x \in (0, \frac{1}{2}]$, then $f(x) = 2x(1 - x) \geq x$, and $f(x) \in (0, \frac{1}{2}]$. So, by induction,

$f^n(x)$ is increasing and bounded above by $\frac{1}{2}$, therefore it approaches a limit, which must be a fixed point, which must be $\frac{1}{2}$ since $x > 0$. Now, note that for any $x \in (0,1)$, $f(x) \in (0, \frac{1}{2}]$, and so by the previous work, $f^n(f(x)) \to \frac{1}{2}$. But this clearly implies that $f^{n+1}(x) \to \frac{1}{2}$, and so $f^n(x) \to \frac{1}{2}$ as well.

2. (This proof is slightly informal.) We claim that for every n, f^n has $2^n + 1$ values of x for which $f^n(x) = 0$, and 2^n values of x for which $f^n(x) = 1$, and that these values alternate (i.e. as x increases, $f^n(x)$ achieves 0, then 1, then 0, then 1, etc.).

Suppose that we know this for n and wish to prove it for $n+1$. Denote the set of x where $f^n(x) = 0$ by Z_n, and the set of x where $f^n(x) = 1$ by O_n. By the inductive hypothesis, $|Z_n| = 2^n + 1$, $|O_n| = 2^n$, and these sets "alternate." Now, note that for any $x \in Z_n \cup O_n$, $f^n(x)$ is 0 or 1, and so $f^{n+1}(x) = 0$; in fact clearly these are the only x for which $f^{n+1}(x) = 0$. So, we define $Z_{n+1} = Z_n \cup O_n$, and note that $|Z_{n+1}| = 2^n + 1 + 2^n = 2^{n+1} + 1$, as desired. Also note that if $f^{n+1}(x) = 1$, then $f^n(x) = \frac{1}{2}$. Since Z_n and O_n alternate, any two closest elements of Z_{n+1} will have images under f^n equal to 0 and 1, and so by the IVT there is a number between them where $f^n(x) = \frac{1}{2}$, meaning that $f^{n+1}(x) = 1$. The set of all such numbers will have size one less than Z_{n+1}, so 2^{n+1}; we'll then call this set O_{n+1}, which has the desired size. Also, by the construction, O_{n+1} and Z_{n+1} "alternate," so by induction we are finished.

Now, since f is continuous, f^n is as well. Therefore, for every pair $z \in Z_n$ and $o \in O_n$, if we consider $i(x) = x$, then $f^n(z) = 0 < i(z)$ and $f^n(o) = 1 > i(o)$. Therefore, by the IVT lemma proved in class, there exists x between z, o so that $f^n(x) = i(x) = x$. By the structure of Z_n and O_n given above, we can partition $[0,1]$ into 2^n intervals with endpoints in Z_n and O_n, and so there are 2^n values of x where $f^n(x) = x$, all of which are periodic with period n by definition.

Written Problem 1: Note that by the Chain Rule,

$$\begin{aligned}(f^n)' &= (f(f^{n-1}))' = f'(f^{n-1})(f^{n-1})' = f'(f^{n-1})(f(f^{n-1}))' \\ &= f'(f^{n-1})f'(f^{n-2})(f^{n-2})' = \cdots = f'(f^{n-1})f'(f^{n-2})\cdots f'(f)f'.\end{aligned}$$

Therefore, if p is periodic with period k, then for any i,

$$\begin{aligned}(f^k)'(f^i p) &= f'(f^{k-1}(f^i p))f'(f^{k-2}(f^i p))\cdots f'(f(f^i p))f'(f^i p) \\ &= f'(f^{k+i-1}p)f'(f^{k+i-2}p)\cdots f'(f^k p)f'(f^{k-1}p)\cdots f'(f^i p) \\ &= f'(f^{i-1}p)f'(f^{i-2}p)\cdots f'(p)f'(f^{k-1}p)\cdots f'(f^i p).\end{aligned}$$

Since multiplication is commutative, this is equal regardless of the value of i; it is always just the product of the values of $f'(x)$ over the values $x = p, f(p), \ldots, f^{k-1}p$ in the orbit of p.

Written Problem 2: Choose any $\mu \in (0,1)$, and define $f_\mu(x) = \mu x(1-x)$, with fixed points $x = 0$ and $x = p_\mu = 1 - \frac{1}{\mu} < 0$. As usual, define $\tilde{p_\mu} = \frac{1}{\mu}$, and

$f_\mu(\tilde{p_\mu}) = p_\mu$. Choose any $x \neq 0, p_\mu, \tilde{p_\mu}$. Clearly the orbit of 0 approaches 0, the orbit of 1 approaches 0, the orbit of p_μ approaches p_μ, and the orbit of $\tilde{p_\mu}$ approaches p_μ. For all other x, we break into cases.

Case 1: $x \in (-\infty, p_\mu)$. Then $1 - x > 1 - p_\mu = \frac{1}{\mu}$, so $f_\mu(x) = \mu x(1-x) < x$. (This is because $\mu(1-x) > 1$, and $x < 0$. In addition, $f_\mu(x) < x < p_\mu$, so $f_\mu(x) \in (-\infty, p_\mu)$ as well. As usual, this implies by induction that the orbit of x is decreasing. If it were bounded from below, it would have to approach a limit less than p_μ. But there are no fixed points less than p_μ, so this cannot happen, meaning that in this case the orbit of x approaches $-\infty$.

Case 2: $x \in (p_\mu, 0)$. Then $1 - x < 1 - p_\mu = \frac{1}{\mu}$, so as shown above, $f_\mu(x) > x$. In addition, x is negative and μ and $1 - x$ are positive, so $f_\mu(x) < 0$, meaning that $p_\mu < x < f_\mu(x) < 0$. So, again as usual, by induction the orbit of x is increasing. It is bounded from above by 0, so it must approach a limit, which must be a fixed point. x is greater than p_μ and the orbit is increasing, so the limit must be greater than p_μ, and the only possible value is 0. So, in this case the orbit of x approaches 0.

Case 3: $x \in (0, 1)$. Then μ and $1 - x$ are in $(0, 1)$, meaning that $f_\mu(x) = \mu x(1-x) < x$. Clearly $f_\mu(x)$ is also positive, so $0 < f_\mu(x) < x < 1$. Again, the usual induction shows that the orbit of x is decreasing and bounded from above by 0, so approaches a nonnegative fixed point limit, and the only choice is 0. So, in this case the orbit of x approaches 0.

Case 4: $x \in (1, \tilde{p_\mu})$. Since f is decreasing for $x > 1$, $f(1) = 0$, and $f(\tilde{p_\mu}) = p_\mu$, we know that $f_\mu(x) \in (p_\mu, 0)$. Then, by Case 2, the orbit of $f_\mu(x)$ approaches 0, meaning that the orbit of x approaches 0 as well.

Case 5: $x \in (\tilde{p_\mu}, \infty)$. Then, since f is decreasing for $x > 1$ and $f(\tilde{p_\mu}) = p_\mu$, we know that $f_\mu(x) \in (-\infty, p_\mu)$. Then, by Case 1, the orbit of $f_\mu(x)$ approaches $-\infty$, meaning that the orbit of x approaches $-\infty$ as well.

Written Problem 3: We assume that f is a C^1-diffeomorphism. Denote by H the set of hyperbolic fixed points of f, i.e. the set of fixed points x so that $|(f^p)'(x)| \neq 1$.

Now, suppose for a contradiction that x is a non-isolated point of H, i.e. x is the limit of a sequence x_n of fixed points of f, none of which are equal to x. Then, for every $n > 0$, we can choose distinct x_{a_n} and x_{b_n} in $(x - n^{-1}, x + n^{-1})$; without loss of generality let's say $x_{a_n} < x_{b_n}$. By the Mean Value Theorem, since both x_{a_n} and x_{b_n} are fixed, there exists $c_n \in [x_{a_n}, x_{b_n}]$ for which

$$f'(c_n) = \frac{f(x_{b_n}) - f(x_{a_n})}{x_{b_n} - x_{a_n}} = \frac{x_{b_n} - x_{a_n}}{x_{b_n} - x_{a_n}} = 1.$$

Since $x_{a_n}, x_{b_n} \in (x - n^{-1}, x + n^{-1})$, $c_n \in (x - n^{-1}, x + n^{-1})$ as well. However, then the sequence c_n clearly approaches x, and since f' is continuous and $f'(c_n) = 1$ for all n, we can conclude that $f'(x) = 1$. However, this means that x was not hyperbolic, a contradiction. Therefore, all points of H are isolated.