MATH 3451 Homework Assignment 3 solutions

9. We'll do this by induction; our inductive hypothesis is that after stage n, we've removed a total length of $\frac{1}{5}\sum_{i=1}^{n}\left(\frac{4}{5}\right)^{i-1}$, and there are 2^n remaining intervals, each with length $\left(\frac{2}{5}\right)^n$.

The base case is $n = 1$. After stage 1, we've removed a length of $\frac{1}{5}$, and there are 2 remaining intervals, each with length $\frac{2}{5}$.

For the inductive step, assume that after stage n, we've removed a total length of $\frac{1}{5}\sum_{i=1}^{n}\left(\frac{4}{5}\right)^{i-1}$, and there are 2^n remaining intervals, each with length $\left(\frac{2}{5}\right)^n$. Then stage $n+1$ involves removing 2^n intervals, each with length $\frac{1}{5}\left(\frac{2}{5}\right)^n$, so an additional length of $2^n \cdot \frac{1}{5} \cdot \frac{2^n}{5^n} = \frac{4^n}{5^{n+1}}$ is removed, meaning that the total length removed after stage $n+1$ is

$$\frac{1}{5}\sum_{i=1}^{n}\left(\frac{4}{5}\right)^{i-1} + \frac{4^n}{5^{n+1}} = \frac{1}{5}\sum_{i=1}^{n+1}\left(\frac{4}{5}\right)^{i-1}.$$

Also, each of the 2^n intervals present before stage $n+1$ leaves 2 intervals, each with length $\frac{2}{5}\left(\frac{2}{5}\right)^n = \left(\frac{2}{5}\right)^{n+1}$, so after stage $n+1$ there are 2^{n+1} remaining intervals each with length $\left(\frac{2}{5}\right)^{n+1}$, completing the induction.

We note that as $n \to \infty$, the total length removed approaches

$$\frac{1}{5}\sum_{i=1}^{\infty}\left(\frac{4}{5}\right)^{i-1} = \frac{1}{5} \cdot \frac{1}{1-\frac{4}{5}} = 1,$$

and so the middle-fifths Cantor set that remains has 'length zero.'

Written Problem 1: Assume that $\mu > 2 + \sqrt{5}$, and consider an arbitrary periodic point p with period k. Note that if $p \notin \Lambda$, then there exists m so that $f^m(p) \notin [0,1]$, and then by results from class, the orbit of $f^m(p)$ approaches $-\infty$, meaning that p could not possibly have been periodic. Therefore, our periodic point p must be in Λ. Again, by definition, the entire orbit of p must be in Λ (because for every m, the orbit of $f^m(p)$ is part of the orbit of p, which stays in $[0,1]$ forever.)

Recall that we proved in class that when $\mu > 2 + \sqrt{5}$, every $x \in \Lambda$ has $|f'(x)| > 1$. Therefore, since $f^m(p) \in \Lambda$ for all m, we can see that

$$|(f^k)'(p)| = |f'(f^{k-1}p)||f'(f^{k-2}p)| \cdots |f'(f(p))||f'(p)| > 1 \cdot 1 \cdots 1 \cdot 1 = 1.$$

Then, by results shown in class, we know that p is repelling. Since p was an arbitrary periodic point, all periodic points for f_μ are repelling.

Written Problem 2: Assume that $\mu > 5$. Suppose that x and y have symbolic codings $S(x)$ and $S(y)$ with the same first n bits, call them $a_0 a_1 \ldots a_{n-1}$, and assume without loss of generality that $x < y$. Then, x and y are in the interval $I_{a_0 \ldots a_{n-1}}$ as defined in class.

Apply the Mean Value Theorem for the interval $[x, y]$ and the function f^n. Then, there exists $c \in [x, y] \subset I_{a_0 a_1 \ldots a_{n-1}}$ so that $(f^n)'(c) = \frac{f^n(x) - f^n(y)}{x - y}$. Note that since $c \in I_{a_0 \ldots a_{n-1}}$, $f^i(c) \in I_0 \cup I_1$ for all $0 \leq i < n$. By results in class, we know that for all $x \in I_0 \cup I_1$, $|f'(x)| \geq \sqrt{(\mu - 2)^2 - 4} > \sqrt{5} > 2$. Therefore,

$$|(f^n)'(c)| = |f'(f^{n-1}c)||f'(f^{n-2}c)| \cdots |f'(f(c))||f'(c)| > 2 \cdot 2 \cdots 2 \cdot 2 = 2^n.$$

Finally, $f^n(x)$ and $f^n(y)$ are in $[0, 1]$ since $x, y \in \Lambda$, so

$$|(f^n)'(c)| = \frac{|f^n(x) - f^n(y)|}{|x - y|} \Longrightarrow |x - y| = \frac{|f^n(x) - f^n(y)|}{|(f^n)'(c)|} < \frac{1}{2^n} = 2^{-n},$$

completing the proof.

Written Problem 3: Suppose that $\mu > 5$ and that x, y have the same symbolic coding. Then, for every n, the symbolic codings of x and y agree on the first n bits, and by Written Problem 2, $|x - y| < 2^{-n}$. Since this is true for all n, $|x - y| \leq 0$, and so $|x - y| = 0$, meaning that $x = y$. We've shown that two numbers cannot share the same symbolic coding, completing the proof.

Written Problem 4: Suppose that $\mu > 5$. From class, we know that there exists a point $x \in \Lambda$ so that x has the symbolic coding

$$S(x) = .1001000010000001\ldots$$

Then, for every n, $\sigma^{n^2} S(x)$ begins with $2n$ 0s. By results discussed in class,

$$\sigma^{n^2} S(x) = S(f^{n^2} x).$$

Note that since 0 is fixed, $f^n 0 \in I_0$ for all n, and so the symbolic coding of 0 is $S(0) = .000\ldots$.

All of this means that the symbolic coding of $f^{n^2} x$ and the symbolic coding of 0 agree on the first $2n$ bits, meaning by Written problem 2 that $|f^{n^2} x - 0| < 2^{-2n} = 4^{-n}$. This obviously implies that $f^{n^2} x \to 0$.

Also, note that for all m, $\sigma^m S(x) \neq S(0)$, since $S(x)$ has infinitely many 1s. Since $\sigma^m S(x) = S(f^m x)$, by Written problem 3 we can conclude that $f^m x \neq 0$, and since m was arbitrary, 0 is not in the orbit of x.