MATH 3451 Homework Assignment 4 solutions

Section 1.6:

2. If a sequence $s = .s_0s_1s_2\ldots$ has period 3, then

$$s = .s_0s_1s_2\ldots = \sigma^3 s = .s_3s_4s_5\ldots$$

meaning that $s_n = s_{n+3}$ for all n. This means that s repeats a 3-digit word forever. There are 8 such sequences:

$\overline{000} = 000000000\ldots,$
$\overline{001} = 001001001\ldots,$
$\overline{010} = 010010010\ldots,$
$\overline{100} = 100100100\ldots,$
$\overline{011} = 011011011\ldots,$
$\overline{110} = 110110110\ldots,$
$\overline{101} = 101101101\ldots,$
$\overline{111} = 111111111\ldots.$

Points in the same orbit are given the same color in the list.

4(a). If $s \in \Sigma'$, then $s = .s_0s_1s_2\ldots$ does not contain any pair of adjacent 0s. Since every adjacent pair of bits in $\sigma s = .s_1s_2s_3\ldots$ already appeared in s, this means that σs also contains no adjacent 0s and so is also in Σ'. Since this holds for all $s \in \Sigma'$, we've shown that Σ' is preserved by σ.

Suppose that $s^{(n)}$ is a sequence in Σ' which converges to a limit s. Suppose for a contradiction that s contains adjacent 0s, say $s_{m-1} = s_m = 0$. By the definition of convergence, there exists N so that for $n > N$, $s^{(n)}$ and s agree on the first m bits (this can be achieved by forcing their distance to be less than 2^{-m}). But then $s^{(N)}$ contains two adjacent 0s and is not in Σ', a contradiction. Our original assumption was then wrong, and s contains no adjacent 0s and so is in Σ'. Since $s^{(n)}$ was an arbitrary convergent sequence in Σ', we've shown that Σ' is closed.

4(c). We proceed mostly as we did in class, in that we want to list all possible finite words/strings in Σ' in some order. However, if we do this directly, we might accidentally create two 0s in a row, e.g.

$$.0|1|01|10|11|010|011|101|\ldots$$

To fix this, we add a single 1 between every pair of adjacent words, as in

$$s = .01110111011111010101011111011\ldots$$

This sequence cannot have two 0s in a row, since each of the finite words being combined (in black) does not have adjacent 0s and we've placed a 1 (in purple) in between each pair. So, $s \in \Sigma'$. Also, for any $t = .t_0 t_1 t_2 \ldots \in \Sigma'$, and any n, the string $t_0 \ldots t_n$ appears in s, and so there is a shift $\sigma^{k_n} s$ beginning with $t_0 \ldots t_n$. Then, the sequence $s^{(n)} = \sigma^{k_n} s$ agrees with more and more letters on s, and so $s^{(n)} \to t$. Since $t \in \Sigma'$ was arbitrary and since each $s^{(n)}$ is in the orbit of s, we've shown that the orbit of s is dense in Σ'.

6. For any $s \in \Sigma_2$, we claim that $W^s(s)$ consists of all t which differ from s on only finitely many digits. Define the set of such t by $A(s)$; we need to show that $A(s) = W^s(s)$.

$A(s) \subseteq W^s(s)$: if $t \in A(s)$, then there exists N so that $t_n = s_n$ for all $n > N$. But then for any $n > N$, $\sigma^n(s) = \sigma^n(t)$, and so $d(\sigma^n(s), \sigma^n(t)) = 0$. But then clearly, $\lim_{n\to\infty} d(\sigma^n(s), \sigma^n(t)) = 0$, and so $t \in W^s(s)$.

$W^s(s) \subseteq A(s)$: Assume that $t \notin A(s)$. Then, there are infinitely many n so that $t_n \neq s_n$. But then for any such n, $\sigma^n(s)$, $\sigma^n(t)$ start with different digits, and so $d(\sigma^n(s), \sigma^n(t)) \geq 2^0 = 1$. This means that the sequence $d(\sigma^n(s), \sigma^n(t))$ contains infinitely many terms which are at least 1, and so cannot possibly approach 0. Therefore, $t \notin W^s(s)$. We showed that $t \notin A(s) \Longrightarrow t \notin W^s(s)$, and the contrapositive is $t \in W^s(s) \Longrightarrow t \in A(s)$.

Written Problem 1: Fix any $\epsilon > 0$. Then there exists N so that $2^{-N} < \epsilon$. Define $\delta = 2^{-N}$. Assume that $s, t \in \Sigma_2$ have $d(s,t) < \delta = 2^{-N}$. Then, as discussed in class, s, t agree on the first N bits. Define $x = S^{-1}s$ and $y = S^{-1}t$. Then x, y have symbolic codings $S(x) = s$ and $S(y) = t$ which agree on the first N bits, and so by Written Problem 2 of last week, (since we assume $\mu > 5$), $|x - y| < 2^{-N} < \epsilon$.

We've shown that $d(s,t) < \delta \Longrightarrow |S^{-1}s - S^{-1}t| < \epsilon$, proving that S^{-1} is (uniformly) continuous.

Written Problem 2: Let's answer this question for Σ_2 first. If a sequence $s = .s_0 s_1 s_2 \ldots$ has period 4, then s comes from repeating a 4-digit word forever. There are 16 such sequences, coming from the 4-letter words

$$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.$$

But we have to be careful: four of these will give sequences with a smaller period, meaning they don't have least period 4. Namely,

$$.0000\ldots \text{ has period } 1, .0101\ldots \text{ has period } 2, .1010\ldots \text{ has period } 2, .1111\ldots \text{ has period } 1.$$

So, there are 12 remaining sequences in Σ_2 with least period 4. Call the set of these sequences T.

We claim that $t \in \Sigma_2$ has least period 4 for σ iff $S^{-1}t \in \Lambda$ has least period 4 for f. To see this, we note that a point z having least period 4 for a function g just means $g^4 z = z$ and $g^i z \neq z$ for $i = 1, 2, 3$. Then, we can see that

$$t = \sigma^4 t \Longleftrightarrow S^{-1}t = S^{-1}\sigma^4 t \Longleftrightarrow S^{-1}t = f^4 S^{-1}t$$

and, for $i = 1, 2, 3$,

$$t \neq \sigma^i t \Longleftrightarrow S^{-1}t \neq S^{-1}\sigma^i t \Longleftrightarrow S^{-1}t \neq f^i S^{-1}t.$$

(Here, we used the commuting diagram property twice, and the fact that S is a bijection when we applied S^{-1} to both sides and preserved inequality.)

So, indeed S^{-1} preserves points of least period 4, and so $S^{-1}T$ is the set of points of least period 4 for f. Since S^{-1} is a bijection, $S^{-1}T$ also has 12 points, and so there are 12 points of least period 4 for f in Λ.