MATH 3451 Homework Assignment 5 solutions

1. Assume that (X, f^2) is topologically transitive, and consider any nonempty open sets U and V. Since (X, f^2) is topologically transitive, there exists $n \geq 0$ and $x \in U$ so that $(f^2)^n x \in V$. But then $x \in U$, $f^{2n}(x) \in V$, and $2n \geq 0$. Since U, V were arbitrary, we've shown topological transitivity of (X, f).

2. Choose any nonempty open U, V, and W. By topological transitivity, there exists $n > 0$ and $x \in V$ for which $f^n(x) \in W$. This means that for this specific n, $V \cap f^{-n}(W) \neq \varnothing$ (it contains x), and so $V \cap f^{-n}(W)$ is a nonempty open set. We can then use topological transitivity again for the sets U and $V \cap f^{-n}(W)$ and see that there exists $m > 0$ so that $x \in U$ and $f^m(x) \in V \cap f^{-n}(W)$. But this means that $x \in U$, $f^m(x) \in V$, and $f^n(f^m(x)) = f^{m+n}(x) \in W$, completing the proof since clearly $m + n > m$.

3. Suppose that f is increasing on $\mathbb{R}$. We break into cases depending on whether or not f has a fixed point.

Case 1: f increasing, has fixed point Denote a fixed point of f by x. Since f is increasing, for any $y > x$, $f(y) > f(x) = x$, and so by induction $f^n(y) > x$ for all n. This means that if we define $U = (x, x+1)$ and $V = (x-1, x)$, then for any $y \in U$, $f^n(y) > x$, and so $f^n(y) \notin V$, for all n, contradicting topological transitivity.

Case 2: f increasing, has no fixed points Then either $f(x) > x$ for all x or $f(x) < x$ for all x (if there existed y, z so that $f(y) \leq y$ and $f(z) \geq z$, then f would have a fixed point by the IVT applied to $g(x) = f(x) - x$). This means that by induction, either all orbits are increasing forever or all orbits are decreasing forever. In either case we can contradict transitivity; for instance, if all orbits increase forever, then for $U = (0, 1)$ and $V = (-1, 0)$, it is not possible to have $x \in U$ with $f^n(x) \in V$ for some n. Similarly, if all orbits decrease forever, then for $U = (-1, 0)$ and $V = (0, 1)$, it is not possible to have $x \in U$ with $f^n(x) \in V$ for some n.

4(a). Choose any $s \in \Sigma''$ and $n > 0$. If the first n digits of s are $s_1 \ldots s_n$, then we create a periodic point in Σ'' as follows. Choose a to be a digit from $1, 2, 3$ not equal to either s_1 or s_n; we can do this since there are three legal symbols in Σ''. Then, the sequence $t^{(n)} = \overline{s_1 \ldots s_n a} = s_1 \ldots s_n a s_1 \ldots s_n a \ldots$ is in Σ'', and agrees with s on its first n digits. Therefore, the sequence $t^{(n)}$ converges to s, and each is periodic, completing the proof that periodic points of Σ'' are dense.

4(b). Note that $s = .1222222\ldots \in \Sigma'''$, and assume for a contradiction that there exists a sequence $t^{(n)}$ of periodic points in Σ''' converging to s. Then, by definition, there exists N so that $t^{(N)}$ agrees with s on two digits, i.e. $t^{(N)} = .12\ldots$; we refer to $t^{(N)}$ as t for simplicity. By the rules of Σ''', $t_n \geq 2$ for all $n > 2$. However, this contradicts periodicity of t; if $\sigma^k t = t$, then $t_k = 1$, which we've shown cannot happen. This contradiction means that no sequence of periodic points of Σ''' converges to s, and so the set of periodic points is not dense in Σ'''.

5. Suppose that f has a fixed point z with $|f'(z)| < 1$. Then, since f' is continuous, there exists a neighborhood $U = (z-\gamma, z+\gamma)$ so that $|f'(y)| < 1$ for all $y \in U$.

Now, we prove the negation of SDIC, i.e. that $\forall \delta > 0$, $\exists \epsilon > 0$ and x so that $\forall y \in (x-\epsilon, x+\epsilon)$, $\forall n \geq 0$, $d(f^n x, f^n y) \leq \delta$. This is actually simple; just define $\epsilon = \min(\delta, \gamma)$ and $x = z$. Then, $(x-\epsilon, x+\epsilon) = (z-\epsilon, z+\epsilon) \subset U$. Then by the MVT, for any $y \in (z-\epsilon, z+\epsilon)$,

$$\frac{|f(y)-z|}{|y-z|} = |f'(t)|$$

for some $t \in U$. Since $t \in U$, $|f'(t)| < 1$, and so $|f(y)-z| < |y-z|$. This implies that $f(y) \in (z-\epsilon, z+\epsilon)$, and by induction that $f^n(y) \in (z-\epsilon, z+\epsilon)$ for all $n \geq 0$. Therefore, for all n, $|f^n(y) - f^n(z)| = |f^n(y) - z| < \epsilon \leq \delta$, contradicting SDIC.