2015 MATH 3451 Midterm Exam

Instructions: You will have 110 minutes for this exam. You should have 5 problems on 10 pages, worth a total of 100 points. You may use facts proved in class for any problem, as long as you state them clearly. If you are not sure whether you are allowed to use a particular fact for some problem, just ask me! Also please ask me if any questions are unclear in any way.

1. (10 pts. each) For the following functions, find all fixed points of f and describe whether they are attracting, repelling, or neither by any argument you wish.

(a) $f(x) = \frac{1}{2}(x^3 + x)$ on $\mathbb{R}$

(b) $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

2. (10 pts. each) The following questions pertain to $f(x) = 4.5x(1 - x)$, and involve use of the symbolic coding s discussed in class. You do NOT have to prove existence of the symbolic coding or its properties which we proved in class (e.g. that it's a topological conjugacy). Define $\Lambda = \{x \ : \ \forall n \geq 0, f^n(x) \in [0, 1]\}$.

(a) Explain why there are exactly 2 points in Λ which have least period 2 under f.

(b) Explain why for every $x \in \Lambda$, $x \neq \frac{7}{9}$, it is NOT the case that $f^n(x) \to \frac{7}{9}$. (HINT: Begin by finding the symbolic coding of $\frac{7}{9}$, and note that $\frac{7}{9}$ is p_μ for this example.)

3. (a) (10 pts.) If $f(x) = x^3$ and $g(x) = \frac{1}{4}x^3$, show that $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ ARE topologically conjugate by exhibiting a conjugacy $\pi : \mathbb{R} \to \mathbb{R}$.

(b) (10 pts.) If $f(x) = x^3$ and $g(x) = \frac{-1}{4}x^3$, show that $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ are NOT topologically conjugate by any argument you wish.

4. (20 pts.) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, has no fixed points, and $f(0) = 1$, prove that f is NOT topologically transitive. (HINT: Can you say anything about which of $f(x)$ and x is larger on all of $\mathbb{R}$?)

5. Suppose that $f'(x) \in (0, \frac{1}{2})$ for all $x \in [0,1]$, and that $f(x) \in [0,1]$ for all $x \in [0,1]$. Define $I_0 = [0,1]$ and $I_n = f^n([0,1])$ for all $n > 0$, i.e. I_n is the set of all $f^n(x)$ for $x \in [0,1]$. Since the continuous images of intervals are intervals, all I_n are intervals; you may use this fact without proof.

(a) (5 pts.) Prove that the sets I_n are nested, i.e. that $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$

(b) (10 pts.) Use the Mean Value Theorem to prove that the intervals I_n have lengths approaching 0. (HINT: start by proving that the length of I_1 is less than $\frac{1}{2}$!)

(c) (5 pts.) Use (a) and (b) to prove that f has exactly one fixed point on $[0, 1]$.