Putnam Practice Problems
September 12, 2019
(in no particular order)

(1) When the numbers 1 through 64 are placed on the sixty-four squares of a checkerboard, define the *gap* to be the largest difference between two numbers on adjacent squares (two squares are adjacent if they share a side or even a corner). Determine the smallest possible gap among all ways of placing the numbers on the squares.

(2) A *tromino* is a rectangle whose dimensions are 1 cm by 3 cm. Let n be of the form $3k+1$ where k is a positive integer. Consider an $n \times n$ square grid of little squares each 1 cm on a side. How many of the little squares have the following property: if deleted, the rest of the board can be perfectly covered by non-overlapping trominoes?

(3) Which positive integers n have the property that there exist choices of the $\pm$ signs for which
$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm (n-1) \pm n = 0?$$

(4) Given any set S of 2015 points in the plane, show that there exists a line ℓ which passes through exactly one point and has 1007 points on each side.

(5) Find the limit $\displaystyle\lim_{n\to\infty} \frac{1^1+2^2+3^3+\ldots+n^n}{n^n}$.

(6) Notice that the number 78 is a three-digit palindrome when written in base 5, since $(78)_{10} = (303)_5$; it is also a three-digit palindrome when written in base 7, since $(78)_{10} = (141)_7$. Prove that there are infinitely many positive numbers N that are three-digit palindromes to two different bases at the same time.

(7) Let T be a triangle with the following property: if T' is any triangle with the same perimeter and area as T, then T' is actually congruent to T. Show that T is equilateral.

(8) Show that no equilateral triangle in the plane has three vertices all of whose coordinates are integers.