Putnam Practice Problems #2
October 4th, 2019
(in no particular order)

(1) Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Compute the lower left-hand entry of A^{2019}.

(2) If the number 2019 is written as a sum of at least two (and possibly more than two!) positive integers, what is the maximum possible product of these integers?

(3) If every point in the plane is colored either red, yellow, green, or blue, show that some two points are a distance of either 1 or $\sqrt{3}$ apart and have the same color.

(4) Consider the one thousand three-digit numbers, where numbers smaller than 1000 have been padded with zeros on the left to make them three digits long as well:

$$\{000, 001, \ldots, 099, 100, \ldots, 998, 999\}.$$

Show that for ANY three-thousand-digit number N formed by concatenating these thousand three-digit numbers together in ANY order, N is a multiple of 37. (HINT: if you understand the proof of the rule "a number is divisible by 9 if the sum of its digits is divisible by 9," you have a good start for this problem.)

(5) Suppose $f : (a,b) \to \mathbb{R}$ has continuous second derivative, and $f''(x) \neq 0$ for all $x \in (a,b)$. Show that two chords on the graph of $y = f(x)$ cannot bisect each other. (A chord on the graph is a line segment between two points on the graph.)

(6) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function where $f(2018) = 2\pi$, and $|f(x) - f(y)|^2 \leq |x - y|^3$ for all real x, y. Find, with proof, $f(2019)$.

(7) The Fibonacci sequence is defined by $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Show that some Fibonacci number a_n is divisible by 2019.

(8) Evaluate, with proof,

$$\int_{\pi/6}^{\pi/3} \frac{1}{1 + (\tan x)^{2018}}\, dx.$$

(9) Find all solutions to $x^3 + 2y^3 + 4z^3 = 0$ where x, y, z are integers.