Putnam Practice Problems #3
October 29th, 2019
(in no particular order)

(1) Let $\triangle ABC$ be an equilateral triangle with sides of length 1. Define S to be the set of all points D such that *exactly one* of the angles $\angle ADB$, $\angle ADC$, and $\angle BDC$ is obtuse. Compute the area of S.

(2) Let n, A, and B be positive integers satisfying $n^2 < A < B < (n+1)^2$. Prove that AB is not a perfect square.

(3) For any real number α, let $\langle \alpha \rangle$ represent the fractional part of α, i.e. the unique real number in $[0,1)$ so that $\alpha - \langle \alpha \rangle$ is an integer. Prove that for some n,

$$\left| \frac{1}{2012} - \langle \sqrt{n} \rangle \right| < \frac{1}{2019^{2019}}.$$

(4) Isaac and Pierre play the following game: they begin with a pile of 2019 pebbles, and take turns removing either one or two pebbles from the pile. Isaac always starts, and the player who removes the last pebble wins. If both players play as well as possible, which player has a winning strategy?

(5) A bunch of children are playing in groups on a playground (a group might have only one child in it). Every minute, one child leaves their current group and joins a group that has at least as many children as their previous group. Prove that eventually all of the children are playing in one huge group.

(6) Show that there are infinitely many positive integers which are not representable as $x^2 + y^3$ for positive integers x and y.

(7) Choose a set S of points in the plane. Then for any point P in the plane (which may or may not be in S!), a *buddy* is a point $Q \in S$ such that the distance from Q to P is strictly less than the distance from Q to any other point in S. What is the maximum possible number of buddies a point P in the plane can have?

(8) Suppose that x is a real number such that $x + \frac{1}{x}$ is an integer. Prove that $x^n + 1/x^n$ is an integer for every integer n.

(9) Suppose you have four points in the plane with the property that any three of them can be simultaneously covered by a disk of radius 1. Show that all four points can be simultaneously covered by a disk of radius 1.