A RAMSEY SPACE OF INFINITE POLYHEDRA
AND THE RANDOM POLYHEDRON

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ABSTRACT. In this paper we introduce a new topological Ramsey space $\mathcal{P}$
whose elements are infinite ordered polyhedra. The corresponding family $\mathcal{AP}$
of finite approximations can be viewed as a class of finite structures. It
turns out that the closure of $\mathcal{AP}$ under isomorphisms is the class $\mathcal{KP}$ of finite
ordered polyhedra. Following [8], we show that $\mathcal{KP}$ is a Ramsey class. Then,
we prove a universal property for ultrahomogeneous polyhedra and introduce
the (ordered) random polyhedron, and prove that it is the Fraïssé limit of
$\mathcal{KP}$; hence the group of automorphisms of the ordered random polyhedron is
extremely amenable (this fact is deduced from results of [6]). Later, we present
a countably infinite family of topological Ramsey subspaces of $\mathcal{P}$; each one
determines a class of finite ordered structures which turns out to be a Ramsey
class. One of these subspaces is Ellentuck’s space; another one is associated
to the class of finite ordered graphs whose Fraïssé limit is the random graph.
The Fraïssé limits of these classes are not pairwise isomorphic as countable
structures and none of them is isomorphic to the random polyhedron. Finally,
following [6], we calculate the universal minimal flow of the (non ordered)
random polyhedron as well as the universal minimal flows of the (non ordered)
random structures associated to our family of topological Ramsey subspaces
of $\mathcal{P}$.

1. INTRODUCTION

A polyhedron is a geometric object built up through a finite or countable number
of suitable amalgamations of convex hulls of finite sets; polyhedra are generated in
this way by simplexes. Simplicial morphisms are locally linear maps that preserve
vertices. An ordered polyhedron is a polyhedron for which we have imposed a linear
order on the set of its vertices. As we only consider order-preserving morphisms,
ordered polyedra are rigid, i.e., admit no non-trivial automorphisms; this is an easy
consequence of the well order principle. In this paper we define a new topological
Ramsey space (see [11]) whose elements are essentially infinite ordered polyhedra.

The theory of topological Ramsey spaces is developed in [11], and was pioneered by
the work [1] of Ellentuck’s. In Section 2 we will describe the fundamental concepts
of that theory. In Section 3 we will define our new topological Ramsey space $\mathcal{P}$.
The closure under isomorphisms of the corresponding family $\mathcal{AP}$ of finite approximations
(viewed as a class of finite structures) turns out to be the class $\mathcal{KP}$ of finite
ordered polyhedra. In Section 4, following [8], we prove the Ramsey property for
the class $\mathcal{KP}$. We also prove a universal property for ultrahomogeneous polyhedra and show that the automorphism group of the Fraïssé limit of $\mathcal{KP}$ is extremely amenable, following [6]. A description of this Fraïssé limit is given in Section 5; we call it the ordered random polyhedron. In Section 6, we introduce a countable family $\{\mathcal{P}(k)\}_{k>0}$ of topological Ramsey subspaces of $\mathcal{P}$. Each $\mathcal{P}(k)$ determines a class $\mathcal{KP}(k)$ of finite ordered structures which turns out to be a Ramsey class. The automorphism group of its Fraïssé limit is therefore extremely amenable. For instance, $\mathcal{P}(1)$ coincides with Ellentuck’s space (see the definition below). The corresponding Ramsey class is of course the class of finite linearly ordered sets whose Fraïssé limit is $(\mathbb{Q}, \leq)$. On the other hand, the Ramsey class associated to $\mathcal{P}(2)$ is the class of finite ordered graphs whose Fraïssé limit is the ordered random graph. It is worth mentioning that the Fraïssé limits of the classes $\mathcal{KP}(k)$, $k > 0$, are not pairwise isomorphic as countable structures, and none of them is isomorphic to the ordered random polyhedron. Finally, following [6], we calculate the universal minimal flow of the (non ordered) random polyhedron as well as the universal minimal flows of the (non ordered) random structures associated to our family of topological Ramsey subspaces of $\mathcal{P}$.

In brief, we introduce some new topological Ramsey spaces associated to polyhedra and related geometric and combinatorial objects, and study their relation with Ramsey classes of finite (ordered) structures and the automorphism groups of their Fraïssé limits, and their universal minimal flows.

**Notation.** Given a countable set $A$, we will adopt the following notation throughout the paper. Let $A$ be a countable set and $X \subseteq A$; then $|X|$ denotes the cardinality of $X$ and:

- $A[k] = \{ X \subseteq A : |X| = k \}$, for every $k \in \mathbb{N}$.
- $A[\leq k] = \{ X \subseteq A : |X| \leq k \}$, for every $k \in \mathbb{N}$.
- $A[< \infty] = \{ X \subseteq A : |X| < \infty \}$.
- $A[\infty] = \{ X \subseteq A : |X| = \infty \}$.

2. **Ramsey spaces**

The definitions and results throughout this section are taken from [11].

2.1. **Metrically closed spaces and approximations.** Consider a triplet of the form $(\mathcal{R}, \leq, r)$, where $\mathcal{R}$ is a set, $\leq$ is a quasi order on $\mathcal{R}$ and $r : \mathbb{N} \times \mathcal{R} \to \mathcal{AR}$ is a function with range $\mathcal{AR}$. For every $n \in \mathbb{N}$ and every $A \in \mathcal{R}$, let us write

\[
r_n(A) := r(n, A)
\]

We say that $r_n(A)$ is the $n$th approximation of $A$. We will reserve capital letters $A, B \ldots$ for elements in $\mathcal{R}$ while lowercase letters $a, b \ldots$ will denote elements of $\mathcal{AR}$.

In order to capture the combinatorial structure required to ensure the provability of an Ellentuck type Theorem, some assumptions on $(\mathcal{R}, \leq, r)$ will be imposed. The first is the following:

(A.1)

(A.1.1) For any $A \in \mathcal{R}$, $r_0(A) = \emptyset$.
(A.1.2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n) (r_n(A) \neq r_n(B))$.
(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n) (r_i(A) = r_i(B))$. 


Take the discrete topology on $\mathcal{AR}$ and endow $\mathcal{AR}^\mathbb{N}$ with the product topology; this is the metric space of all the sequences of elements of $\mathcal{AR}$. The set $\mathcal{R}$ can be identified with the corresponding image in $\mathcal{AR}^\mathbb{N}$. We will say that $\mathcal{R}$ is **metrically closed** if, as a subspace $\mathcal{AR}^\mathbb{N}$ with the inherited topology, it is closed. The basic open sets generating the metric topology on $\mathcal{R}$ inherited from the product topology of $\mathcal{AR}^\mathbb{N}$ are of the form:

$$ (2) \quad [a] = \{ B \in \mathcal{R} : (\exists n)(a = r_n(B)) \} $$

where $a \in \mathcal{AR}$.

Let us define the **length** of $a$, as the unique integer $|a| = n$ such that $a = r_n(A)$ for every $n \in \mathbb{N}$, let

$$ (3) \quad \mathcal{AR}_n := \{ a \in \mathcal{AR} : |a| = n \} $$

Hence,

$$ (4) \quad \mathcal{AR} = \bigcup_{n \in \mathbb{N}} \mathcal{AR}_n $$

The **Ellentuck type neighborhoods** are of the form:

$$ (5) \quad [a, A] = \{ B \in \mathcal{R} : (\exists n) a = r_n(B) \land B \leq A \} $$

where $a \in \mathcal{AR}$ and $A \in \mathcal{R}$.

We will use the symbol $[n, A]$ to abbreviate $[r_n(A), A]$.

Let

$$ (6) \quad \mathcal{AR}(A) = \{ a \in \mathcal{AR} : [a, A] \neq \emptyset \} $$

Given a neighborhood $[a, A]$ and $n \geq |a|$, let $r_n[a, A]$ be the image of $[a, A]$ by the function $r_n$, i.e.,

$$ (7) \quad r_n[a, A] = \{ r_n(B) : B \in [a, A] \} $$

2.2. **Ramsey sets.** A set $X \subseteq \mathcal{R}$ is **Ramsey** if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$. A set $X \subseteq \mathcal{R}$ is **Ramsey null** if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap X = \emptyset$.

2.3. **Topological Ramsey spaces.** We say that $(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** iff subsets of $\mathcal{R}$ with the Baire property are Ramsey and meager subsets of $\mathcal{R}$ are Ramsey null.

Given $a, b \in \mathcal{AR}$, write

$$ (8) \quad a \sqsubseteq b \text{ iff } (\exists A \in \mathcal{R}) (\exists m, n \in \mathbb{N}) m \leq n, a = r_m(A) \land b = r_n(A). $$

By A.1, $\sqsubseteq$ can be proven to be a partial order on $\mathcal{AR}$.

(A.2) **[Finitization]** There is a quasi order $\leq_{fin}$ on $\mathcal{AR}$ such that:

(A.2.1) $A \leq B$ iff $(\forall n) (\exists m)(r_n(A) \leq_{fin} r_m(B))$.

(A.2.2) $\{ b \in \mathcal{AR} : b \leq_{fin} a \}$ is finite, for every $a \in \mathcal{AR}$.

(A.2.3) If $a \leq_{fin} b$ and $c \sqsubseteq a$ then there is $d \sqsubseteq b$ such that $c \leq_{fin} d$. 
Given $A \in \mathcal{R}$ and $a \in \mathcal{AR}(A)$, we define the depth of $a$ in $A$ as
\begin{equation}
\text{depth}_A(a) := \min\{n : a \leq_{\text{fin}} r_n(A)\}
\end{equation}

\textbf{(A.3) [Amalgamation]} Given $a$ and $A$ with depth$_A(a) = n$, the following holds:
\begin{enumerate}[leftmargin=*,label=(\text{A.3}.)]
\item \forall B \in [n, A] \ ((a, B) \neq \emptyset).
\item \forall B \in [a, A] \ ((\exists A' \in [n, A]) \ ((a, A') \subseteq [a, B]).
\end{enumerate}

\textbf{(A.4) [Pigeonhole Principle]} Given $a$ and $A$ with depth$_A(a) = n$, for every $O \subseteq \mathcal{AR}_{|a|+1}$ there is $B \in [n, A]$ such that $r_{|a|+1}[a, B] \subseteq O$ or $r_{|a|+1}[a, B] \subseteq O^c$.

\textbf{Theorem 2.3.1} (Todorcevic, [11]). \textbf{[Abstract Ellentuck Theorem]} Any $(\mathcal{R}, \leq, r)$ with $\mathcal{R}$ metrically closed and satisfying (A.1)-(A.4) is a topological Ramsey space.

\section{The topological Ramsey space $\mathcal{P}$}

In this Section we will construct a new Ramsey space $\mathcal{P}$ and the set of its finite approximations $\mathcal{AP}$.

\subsection{Definition of $\mathcal{P}$}

Consider pairs $(x, S_x)$ satisfying the following conditions:
\begin{enumerate}[leftmargin=*,label=(\text{i}.),ref=(\text{i}).]
\item $x \subseteq \mathbb{N}$.
\item $S_x \subseteq \mathbb{N}^{<\omega}$ is hereditary, i.e., $u \subseteq v \& v \in S_x \Rightarrow u \in S_x$, and
\item $\bigcup S_x = \bigcup\{u : u \in S_x\} = x$.
\end{enumerate}

Given two such pairs $(x, S_x), (y, S_y)$ let us define
\begin{equation}
(y, S_y) \leq (x, S_x) \iff y \subseteq x \& S_y \subseteq S_x.
\end{equation}

Let us write $\mathcal{AP}$ (resp. $\mathcal{P}$) for the set of all pairs $(x, S_x)$ satisfying properties (i), (ii), (iii) and such that $x$ is a finite (resp. an infinite) subset of $\mathbb{N}$. From now on, the elements of $\mathcal{P}$ will be written $(A, S_A), (B, S_B), \ldots$, using capital letters. Let us define the preorder $\leq_{\text{fin}}$ on $\mathcal{AP}$ as follows:
\begin{enumerate}[leftmargin=*,label=\text{(\text{A}.\text{\text{A}}).}]
\item $(a, S_a) \leq_{\text{fin}} (b, S_b) \iff (a, S_a) \leq (b, S_b) \& \max(a) = \max(b)$
\item $(a, S_a) \subseteq (b, S_b) \iff a \subseteq b \& (a, S_a) \leq (b, S_b)$
\end{enumerate}

Here we are using the same symbol $\subseteq$ to indicate that the set $a$ is an initial segment of the set $b$.

Given a pair $(A, S_A) \in \mathcal{P}$ and any subset $x \subseteq A$ (finite or countable), let $S_A \upharpoonright x = \{u \cap x : u \in S_A\}$. In particular, if $n \in \mathbb{N}$ let $A \upharpoonright n$ be the set of the first $n$ elements of $A$ and $S_{A\upharpoonright n} = S_A \upharpoonright (A \upharpoonright n)$. The pair
\begin{equation}
r_n(A, S_A) = (A \upharpoonright n, S_{A\upharpoonright n})
\end{equation}

is the $n$th approximation of $(A, S_A)$. Notice that
\begin{equation}
i \leq j \Rightarrow r_i(A, S_A) \leq r_j(A, S_A) \leq (A, S_A) \quad \forall i, j \in \mathbb{N}
\end{equation}

There is a well defined surjective function
\begin{equation}
\mathcal{P} \times \mathbb{N} \xrightarrow{r} \mathcal{AP} \quad \text{and} \quad r((A, S_A), n) = r_n(A, S_A)
\end{equation}
3.2. \( \mathcal{P} \) is a topological Ramsey space. In the rest of this section we shall prove the following:

**Theorem 3.2.1.** \( (\mathcal{P}, \leq, r) \) is a topological Ramsey space.

The proof of Theorem 3.2.1 will be divided into several lemmas, showing that \( (\mathcal{P}, \leq, r) \) satisfies the conditions of the Abstract Ellentuck Theorem.

**Lemma 3.2.2.** \( (\mathcal{P}, \leq, r) \) satisfies axiom A.1

1. For every \( (A, S_A) \in \mathcal{P} \), \( r_0(A, S_A) = \emptyset \).
2. If \( (A, S_A) \neq (B, S_B) \) then there exists \( n \) such that \( r_n(A, S_A) \neq r_n(B, S_B) \).
3. If \( r_n(A, S_A) = r_m(B, S_B) \) then \( n = m \) and for every \( i < n \), \( r_i(A, S_A) = r_i(B, S_B) \).

**Proof:** Straightforward. \( \square \)

Hence each element of \( \mathcal{P} \) can be identified with the sequence of its approximations.

Next we consider \( \mathcal{P} \) as a subset of the product space \( \mathcal{AP}^\mathbb{N} \), regarding \( \mathcal{AP} \) as a discrete space.

**Lemma 3.2.3.** \( \mathcal{P} \) is a closed subset of \( \mathcal{AP}^\mathbb{N} \).

**Proof:** Consider the injection \( \varphi : \mathcal{P} \rightarrow \mathcal{AP}^\mathbb{N} \) given by

\[
\varphi(A, S_A) = (r_0(A, S_A), r_1(A, S_A), \ldots).
\]

Let us show that \( \varphi(\mathcal{P}) \) is closed. Given a closure point \( \alpha = \{(a^j, S_{a^j})\}_{j \in \mathbb{N}} \) in \( \varphi(\mathcal{P}) \subset \mathcal{AP}^\mathbb{N} \) and a sequence \( \{(A^k, S_{A^k})\}_{k \in \mathbb{N}} \) in \( \mathcal{P} \), if \( \{\varphi(A^k, S_{A^k})\}_{k \in \mathbb{N}} \) converges to \( \alpha \) then

\[
(\forall n \in \mathbb{N}) (\exists k_n \in \mathbb{N}) k \geq k_n \Rightarrow (\forall j \leq n) r_j(A^k, S_{A^k}) = (a^j, S_{a^j}).
\]

Taking a strictly increasing sequence \( k_n < k_{n+1} \forall n \in \mathbb{N} \) we get

\[
n > m \Rightarrow r_m(A^{k_n}, S_{A^{k_n}}) = (a^{k_n}, S_{a^{k_n}}).
\]

Define \( A = \bigcup_{n \in \mathbb{N}} a^{k_n} \) and \( S_A = \bigcup_{n \in \mathbb{N}} S_{a^{k_n}} \). Then \( (A, S_A) \in \mathcal{P} \) and \( \varphi(A, S_A) = \alpha \) by construction. \( \square \)

The following two lemmas are straightforward; we leave the details to the reader.

**Lemma 3.2.4.** \( (\mathcal{P}, \leq, r) \) satisfies axiom A.2

1. If \( (A, S_A) \leq (B, S_B) \) then \( \forall n \exists m, r_n(A, S_A) \leq_{fin} r_m(B, S_B) \).
2. For every \( (a, S_a) \in \mathcal{AP} \) the set \( \{(b, S_b) : (b, S_b) \leq_{fin} (a, S_a)\} \) is finite.
3. If \( (a, S_a) \leq_{fin} (b, S_b) \) and \( (c, S_c) \subseteq (a, S_a) \) then there is \( (d, S_d) \subseteq (b, S_b) \) such that \( (c, S_c) \leq_{fin} (d, S_d) \).

Before stating Lemma 3.2.5 below, let us adapt from Section 2.3 the definiton of basic open sets, for the Ellentuck-like topology of \( \mathcal{P} \). These will be sets \( [(a, S_a), (A, S_A)] \) such that \( (B, S_B) \in [(a, S_a), (A, S_A)] \) iff

\[
(B, S_B) \leq (A, S_A) \& (\exists n) r_n(B, S_B) = (a, S_a)
\]

In particular,

\[
[n, (A, S_A)] = [r_n(A, S_A), (A, S_A)]
\]
For \((a, S_a) \in \mathcal{AP}\) and \((A, S_A) \in \mathcal{P}\) such that \([(a, S_a), (A, S_A)] \neq \emptyset\), let us adapt the definition of depth of \((a, S_a)\) in \((A, S_A)\) as follows (cf. Axiom A.2, p.4):

\[
\text{depth}_{(A, S_A)}(a, S_a) := \min\{n : (a, S_a) \leq_{fin} r_n(A, S_A)\}.
\]

**Lemma 3.2.5.** \((\mathcal{P}, \leq, r)\) satisfies axiom A.3

Let \(n = \text{depth}_{(B, S_B)}(a, S_a)\).

1. If \([(a, S_a) \in [n, (B, S_B)]\) then \([(a, S_a), (A, S_A)] \neq \emptyset\).
2. For every \((A, S_A) \in [(a, S_a), (B, S_B)]\) there exists \((A', S_{A'}) \in [n, (B, S_B)]\) such that \(\emptyset \neq [(a, S_a), (A', S_{A'})] \subseteq [(a, S_a), (A, S_A)]\).

Given \(n \in \mathbb{N}\) let

\[
\mathcal{AP}_n := \{(a, S_a) \in \mathcal{AP} : |a| = n\}
\]

If \((a, S_a) \in \mathcal{AP}_n\) we say that the length of \((a, S_a)\) is \(n\) or simply write \(|(a, S_a)| = n\).

Also, as in the general setting, for every natural number \(n\) write

\[
r_n[(a, S_a), (A, S_A)] = \{r_n(B, S_B) : (B, S_B) \in [(a, S_a), (A, S_A)]\}
\]

Finally, we prove the following

**Lemma 3.2.6.** Pigeonhole principle A.4 for \((\mathcal{P}, \leq, r)\):

Let \(n = \text{depth}_{(B, S_B)}(a, S_a), k = |(a, S_a)|\) and \(c : \mathcal{AP}_{k+1} \to \{0, 1\}\) be any partition. There exists \((A, S_A) \in [n, (B, S_B)]\) such that \(c\) is constant in \(r_{k+1}[(a, S_a), (A, S_A)]\).

**Proof:** Let

\[X = \{m \in B : m > \max(a)\}.
\]

For \(i \in \{0, 1\}\), let

\[X_i = \{m \in X : c((a \cup \{m\}, S_B \upharpoonright a \cup \{m\})) = i\}.
\]

By the classical pigeonhole principle, there is \(i_0 \in \{0, 1\}\) such that \(|X_{i_0}| = \infty\). So let

\[A = (B \upharpoonright n) \cup X_{i_0}\]

and \(S_A = S_B \upharpoonright A\)

Then \((A, S_A) \in [n, (B, S_B)]\) is as required. \(\square\)

Now we can prove that \((\mathcal{P}, \leq, r)\) is a topological Ramsey space:

**Proof of Theorem 3.2.1.** In virtue of the abstract Ellentuck theorem, the required result follows from Lemmas 3.2.2, 3.2.3, 3.2.4, 3.2.5, 3.2.6. \(\square\)

**Remark 3.2.7.** (Ellentuck’s space as a subspace of \(\mathcal{P}\)) Notice that we can identify each \(A \in \mathbb{N}[\infty]\) with the pair \((A, A[\leq 1])\). In this way, we can view \(\mathbb{N}[\infty]\) as a closed subspace of \(\mathcal{P}\).

Recall the approximation function \(i : \mathbb{N} \times \mathbb{N}[\infty] \to \mathbb{N}[<\infty]\), given by

\[i(n, A) = \text{the first } n \text{ elements of } A.
\]

Let \(\mathcal{E} = (\mathbb{N}[\infty], \subseteq, i)\), where \(\subseteq\) is the inclusion relation and \(i\) is the approximation function defined above. For the space \(\mathcal{E}\), the set of approximations is \(\mathcal{AE} = \mathbb{N}[<\infty]\).

For every \(a, b \in \mathbb{N}[<\infty]\), \(a \leq_{fin} b\) if and only if \(a \subseteq b\) & \(\max(a) = \max(b)\). Now we give an alternative proof to the well known fact that \(\mathcal{E}\) is a topological Ramsey space.
Corollary 3.2.8. (Ellentuck [1], 1974) \( \mathcal{E} = (\mathbb{N}^{[\infty]}, \subseteq, i) \) is a topological Ramsey space.

Proof: Fix \( \mathcal{X} \subseteq \mathbb{N}^{[\infty]} \) with the Baire property with respect to the exponential topology of \( \mathcal{E} \). Since \( \mathcal{E} \) is a closed subspace of \( \mathcal{P} \), it is easy to show that the set

\[ \mathcal{X}' = \{(A, A^{[\leq 1]}): A \in \mathcal{X}\} \subset \mathcal{P} \]

has the Baire property with respect to the Ellentuck-like topology of \( \mathcal{P} \). Given a nonempty neighborhood \( [a, A] \) in \( \mathcal{E} \), let \( S_a = a^{[\leq 1]} \) and \( S_A = A^{[\leq 1]} \). Then, consider the neighborhood \( [(a, S_a), (A, S_A)] \) in \( \mathcal{P} \). Applying Theorem 3.2.1 we obtain \( (B, S_B) \in [(a, S_a), (A, S_A)] \) such that

\[ [(a, S_a), (B, S_B)] \subseteq \mathcal{X}' \text{ or } [(a, S_a), (B, S_B)] \cap \mathcal{X}' = \emptyset. \]

Notice that, by necessity, \( S_B = B^{[\leq 1]} \). Hence, \( [a, B] \subseteq \mathcal{X} \) or \( [a, B] \cap \mathcal{X} = \emptyset \).

If \( \mathcal{X} \) is meager with respect to the exponential topology of \( \mathcal{E} \) then the same argument works but in addition the case \( [a, B] \subseteq \mathcal{X} \) will never happen, by the meagerness of \( \mathcal{X} \). This completes the proof. □

From now on, we will refer to \( \mathcal{E} = (\mathbb{N}^{[\infty]}, \subseteq, i) \) as Ellentuck’s space.

3.3. Embeddings of ordered polyhedra. A finite ordered polyhedron is a finite geometric polyhedron for which we have prefixed a linear order on the set of its vertices; it corresponds to a pair \( (x, S_x) \in \mathcal{AP} \) considering \( x \) with the natural order of \( \mathbb{N} \). Hence, \( \mathcal{AP} \) can be understood as a subclass (in the sense of [6], for instance) of the class of finite ordered polyhedra. Similarly, \( \mathcal{P} \) can be understood as a set of ordered polyhedra with a countable set of vertices. We call it the Ramsey space of infinite countable ordered polyhedra. The following will be useful to study this objects in relation to structural Ramsey theory. An embedding

\[ (x, S_x) \xrightarrow{f} (y, S_y) \]

is an injective function \( x \xrightarrow{f} y \) such that \( u \in S_x \Rightarrow f(u) \in S_y \). It is a strong embedding if \( u \in S_x \Leftrightarrow f(u) \in S_y \). A rigid embedding is a strong embedding \( (x, S_x) \xrightarrow{f} (y, S_y) \) such that \( f \) is order-preserving: \( i < j \Rightarrow f(i) < f(j) \).

Lemma 3.3.1. Each finite ordered polyhedron can be embedded in some \( n \)-simplex \( \Delta \), for some \( n \in \mathbb{N} \), and rigidly embedded in some subpolyhedron of \( \Delta \).

Proof: If \( (x, S_x) \in \mathcal{AP} \) is a finite ordered polyhedron let \( n = |x| \), the cardinality of \( x \), and write \( x = \{x_0, x_1, \ldots, x_{n-1}\} \) in increasing order. Define \( x \xrightarrow{f} n \) by \( f(x_j) = j \), for \( j < n \). Then, \( f \) induces an embedding \( (x, S_x) \xrightarrow{f} (n, 2^n) \) which, geometrically, is just the embedding of the polyhedron \( K \) determined by \( (x, S_x) \).

1This can be deduced from two facts: (a) Since \( \mathcal{E} \) is closed in \( \mathcal{P} \), every meager subset of \( \mathcal{E} \) is still meager in \( \mathcal{P} \); and (b) Subsets of \( \mathcal{P} \) with the Baire property form a \( \sigma \)-algebra.
in the standard \( n \)-simplex \( \Delta^n \subset \mathbb{R}^{n+1} \). On the other hand, \( f \) also induces a rigid embedding \( (x, S_x) \xrightarrow{f} (n, S_n) \) where \( S_n = \{ u \subseteq n : \{ x_j : j \in u \} \in S_x \} \).

4. Finite polyhedra as a Ramsey class

In this section we describe some basic concepts and results on Ramsey classes, Fraïssé theory and extremely amenability of automorphism groups, and prove that the class of finite ordered polyhedra is Ramsey. We prove that the automorphism group of its Fraïssé limit is extremely amenable, and state a universal property for ultrahomogeneous polyhedra.

For the rest of this article, we will consider \( L \)-structures \( A = \langle A, C^A, R^A, F^A \rangle \) on a fixed (first order) signature \( L = \langle C, R, F \rangle \) of constants, relations, and functions symbols. Definitions such as morphisms, embeddings, isomorphisms, automorphisms, substructures, etc., can be found in the classical literature (for instance, see [5]).

4.1. Basic concepts. The age of an \( L \)-structure \( A \) is the class \( \text{Age}(A) \) of all finite \( L \)-structures which are isomorphic to some substructure of \( A \). A structure \( F \) is ultrahomogeneous iff each isomorphism between any two finite substructures of \( F \) can be extended to some automorphism of \( F \). A Fraïssé structure is a countable, locally finite, ultrahomogeneous structure.

**Theorem 4.1.1** (Fraïssé). Any two (infinite) countable ultrahomogeneous \( L \)-structures having the same age are isomorphic.

**Theorem 4.1.2.** A non empty class of finite \( L \)-structures \( C \) is the age of a Fraïssé structure iff it satisfies:

1. \( C \) is closed under isomorphisms: If \( A \in C \) and \( A \cong B \) then \( B \in C \).
2. \( C \) is hereditary: If \( A \in C \) and \( B \leq A \) then \( B \in C \).
3. \( C \) contains structures with arbitrarily high finite cardinality.
4. Joint embedding property: If \( A, B \in C \) then there is \( D \in C \) such that \( A \leq D \) and \( B \leq D \).
5. Amalgamation property: Given \( A, B_1, B_2 \in C \) and embeddings \( A \xrightarrow{f_i} B_i \), \( i \in \{1, 2\} \), there is \( D \in C \) and embeddings \( B_i \xrightarrow{g_i} D \) such that \( g_1 \circ f_1 = g_2 \circ f_2 \).

In such case, \( C \) is said to be a Fraïssé class, and there exists a unique (up to isomorphism) countable Fraïssé structure \( F \) such that \( \text{Age}(F) = C \); this \( F \) is the Fraïssé limit of \( C \) and we write \( F = \text{FLim}(C) \).

4.2. Ramsey classes of structures. Given \( L \)-structures \( A, B, C \) we write \( \begin{pmatrix} B \\ A \end{pmatrix} \) for the set of substructures of \( B \) which are isomorphic to \( A \). Given an integer \( r > 0 \), if \( A \leq B \leq C \) then we write \( C \xrightarrow{r} (B)^A \) whenever for each \( r \)-coloring

\[
c : \begin{pmatrix} C \\ A \end{pmatrix} \xrightarrow{r}
\]

of the set \( \begin{pmatrix} C \\ A \end{pmatrix} \), there exists \( B' \in \begin{pmatrix} C \\ B \end{pmatrix} \) such that \( \begin{pmatrix} B' \\ A \end{pmatrix} \) is monochromatic.
A Fraïssé class $C$ has the **Ramsey property** iff, for every integer $r > 1$ and every $A, B \in C$ such that $A \leq B$, there is $C \in C$ such that

$$C \longrightarrow (B)^A_r$$

Also, remember that a topological group $G$ is **extremely amenable** or has the **fixed point on compacta property**, if for every continuous action of $G$ on a compact space $X$ there exists $x \in X$ such that for every $g \in G$, $g \cdot x = x$. If $G$ is an extremely amenable group, then its universal minimal flow is a singleton, a fact that is a remarkable result in Topological Dynamics. The following is an important characterization of the type of groups.

**Theorem 4.2.1.** [6] Let $\mathbb{F}$ be a Fraïssé structure and $C = \text{Age}(\mathbb{F})$. The polish group $\text{Aut}(\mathbb{F})$ is extremely amenable if and only if $C$ has the Ramsey property and all the structures of $C$ are rigid.

### 4.3. Finite polyhedra as a Ramsey class.

Consider $L = \langle (R_i)_{i \in \mathbb{N}\setminus\{0\}} \rangle$, a signature with an infinite number of relational symbols such that for each $i \in \mathbb{N}$ the arity of $R_i$ is $n(i) = i$.

A **polyhedron** is a countable $L$-structure $A = \langle A, (R_i^A)_{i \in \mathbb{N}\setminus\{0\}} \rangle$ such that for each $\{a_1, \ldots, a_i\} \subseteq A$, $(a_1, \ldots, a_i) \in R_i$ if and only if $(a_{\sigma(1)}, \ldots, a_{\sigma(i)}) \in R_i$, for every permutation $\sigma$ of the set $\{1, \ldots, i\}$. Also, if $(a_1, \ldots, a_i) \in R_i$, then for every $k \leq i$ and every subset $\{a_{j_1}, \ldots, a_{j_k}\}$, we have $(a_{j_1}, \ldots, a_{j_k}) \in R_k$. Notice that if $A$ is a finite $L$-structure, then there is a maximum arity $n = n(A)$ such that $R_m^A \neq \emptyset$ and $R_m^A = \emptyset$, for every $m > n$. An **ordered polyhedron** is a $L \cup \{\prec\}$-structure $A = \langle A, (R_i^A)_{i \in \mathbb{N}\setminus\{0\}}, \prec^A \rangle$ such that $\langle A, (R_i^A)_{i \in \mathbb{N}\setminus\{0\}} \rangle$ is a polyhedron and $\prec^A$ is a total ordering on $A$.

Let $\mathcal{K}P_0$ be the class of finite polyhedra and $\mathcal{K}P$ the class of finite ordered polyhedra. It is easy to see that each pair $(x,S_x) \in \mathcal{P} \cup \mathcal{AP}$ is a countable $L \cup \{\prec\}$-structure whose universe is $x$ and in which $S_x$ is a countable family of relations over $x$. The notions of substructure, homorphism, etc, are induced by the embeddings defined in Section 3.3. Furthermore, each one of these structures is rigid (ordered) by construction.

**Remark 4.3.1.** The following facts are straightforward:

- $\mathcal{AP} \subseteq \mathcal{K}P$.
- For every $A \in \mathcal{K}P$ there is $(a, S_a) \in \mathcal{AP}$ such that $A \cong (a, S_a)$. Actually, $\mathcal{K}P$ is the closure of $\mathcal{AP}$ under isomorphisms.

We shall prove that the class $\mathcal{K}P$ is Ramsey in Theorem 4.3.2 below. Before doing that we borrow the notation of [8]: Let $\Delta = \{n_i\}_{i \in I}$ be a finite family of natural numbers. A **set system of type** $\Delta$ is a structure $(X, \leq_X, \mathcal{M})$ such that $(X, \leq_X)$ is a totally ordered set and $\mathcal{M} = \{M_i\}_{i \in I}$ is such that $M \in X^{[n_i]}$, for every $M \in \mathcal{M}_i$. Given two set systems of type $\Delta$, $(X, \leq_X, \mathcal{M})$ and $(Y, \leq_Y, \mathcal{N})$ (with $\mathcal{N} = \{N_i\}_{i \in I}$), we say that $(X, \leq_X, \mathcal{M})$ is a **subobject** of $(Y, \leq_Y, \mathcal{N})$ whenever

- $X \subseteq Y$,
- $\leq_Y | X \times X = \leq_X$ and
- $M_i = \{M \in N_i : M \subseteq X\}$. 


Theorem A in [8] implies in particular that, for a fixed \( \Delta \), the class of all sets systems of type \( \Delta \) (together with all the embeddings) is Ramsey. It is easy to see that each \( A \in \mathcal{KP} \) is a set system of some type \( \Delta_A \). Actually, if \( n = n(A) \) is the maximum arity in \( A \) then \( \Delta_A = \{ i \}^{\leq n} \).

**Theorem 4.3.2.** The class \( \mathcal{KP} \) of all finite ordered polyhedra is Ramsey.

*Proof.* This follows as an application of Theorem A in [8].

Let \( A \leq B \in \mathcal{KP} \) be given. Notice that \( \Delta_A \) is an initial segment of \( \Delta_B \), so we can assume that \( A \) and \( B \) have the same type (some relations in \( A \) can be empty). By Theorem A, there exists a set system \( C = (X, \leq_X, M), M = \{ M_i \}_{i \in I}, \) such that

- \( \Delta_C = \Delta_B \), and
- for every \( r > 1 \), \( C \rightarrow (B)^A_r \).

Set \( S_X = \{ u \in \bigcup_{i \in I} M_i : u \) is a face a copy of \( B \) inside \( C \} \cup X^{[\leq 1]} \) and let \( D = (X, \leq_X, S_X) \). Then, \( D \in \mathcal{KP} \) and for every \( r > 1 \), \( D \rightarrow (B)^A_r \). This completes the proof. \( \square \)

**Remark 4.3.3.** The class of finite ordered polyhedra \( \mathcal{KP} \) satisfies conditions (1), ..., (5) of Theorem 4.1.2; so it is the age of a Fraïssé structure. Let \( P = \text{FLim}(\mathcal{KP}) \), the Fraïssé limit of \( \mathcal{KP} \).

**Corollary 4.3.4.** The automorphism group of \( \text{FLim}(\mathcal{KP}) \) is extremely amenable.

4.4. Geometric characterization of \( P = \text{FLim}(\mathcal{KP}) \). Now we will provide some arguments which are similar to those which arise in the construction of the Fraïssé limit of the class \( \mathcal{KG} \) of finite graphs; say \( \Gamma = \text{FLim}(\mathcal{KG}) \). There is a geometric characterization of \( \Gamma \). For each countable graph \( G = (V, E) \); we have \( G \cong \Gamma \) iff the following holds: For any finite disjoint subsets of vertices \( x, y \subset V \), there is some vertex \( q \in V \setminus (x \cup y) \) such that \( q \) is adjacent to all elements in \( x \) and to none in \( y \). See [5, p.336-337]. In order to show an analogous statement for \( P \), we will start with two simple observations.

Given a finite polyhedron \( (a, S_a) \), we say that \( T \subset P(a) \) generates \( S_a \) if \( S_a = \{ u : \exists v \in T(u \subseteq v) \} \). The \( \subseteq \)-minimal family generating \( S_a \) is \( T_a = \max(S_a) \), the set of maximal subsets of \( a \) in \( S_a \) with respect to \( \subseteq \). Geometrically speaking, for any \( T \) generating \( S_a \), \( T \) is a set of simplexes whose amalgamation (union) in \( \mathbb{R}^{[a]+1} \) is the (geometric realization of the) polyhedron \( (a, S_a) \); and \( T_a \) is the family of maximal subsimplexes of \( (a, S_a) \).

A **one-point extension** of a finite polyhedron \( (a, S_a) \) is a finite polyhedron \( (b, S_b) \) such that \( (a, S_a) \leq (b, S_b) \) and \( b = a \cup \{ p \} \) for some \( p \notin a \). Then \( p \) determines a partition of \( T_a \) into two classes: those \( u \in T_a \) such that \( u \cup \{ p \} \in S_b \), and the other ones.

**Lemma 4.4.1.** For any countable polyhedron \( A = (A, S_A) \), \( A \) is ultrahomogeneous iff the following condition holds:

\((*)\) For each finite polyhedron \( (a, S_a) \), each embedding \( (a, S_a) \xrightarrow{f} A \), and each one-point extension \( (b, S_b) \xrightarrow{g} (a, S_a) \); there exists an embedding \( (b, S_b) \xrightarrow{g} A \) such that \( g \upharpoonright a = f \).
Proof: The direct implication is trivial. For the reciprocal, apply induction. □

Proposition 4.4.2. (Universal property for ultrahomogeneous polyhedra)
A countable polyhedron \( \mathcal{A} = (A, S_A) \) is ultrahomogeneous if and only if for any finite non-empty disjoint subsets \( x, y \subseteq S_A \), such that the elements of \( x \cup y \) are not comparable by \( \subseteq \), there is some vertex \( q \not\in \cup(x \cup y) \) of \( \mathcal{A} \) such that \( u \cup \{q\} \in S_A \ \forall u \in x \), and \( u \cup \{q\} \not\in S_A \ \forall u \in y \).

Proof: Fix \( \mathcal{A} = (A, S_A) \). Let us show that the above geometric condition is equivalent to Condition (*) in Lemma 4.4.1.

\[ (\Rightarrow) \text{ Given a finite polyhedron } (a, S_a), \text{ an embedding } (a, S_a) \xrightarrow{f} \mathcal{A}, \text{ and a one-point extension } (b, S_b) > (a, S_a) \text{ with } b = a \cup \{p\}; \text{ consider the partition } \]
\[ x_0 = \{v \in T_a : v \cup \{p\} \in T_b\} \quad y_0 = \{v \in T_a : v \cup \{p\} \not\in T_b\} \]
\[ \text{of } T_a. \text{ Take } x = f(x_0), \text{ and } y = f(y_0), \text{ so } x \cup y = f(T_a). \text{ By our assumption; there is } q \in A \setminus f(\cup T_a) \text{ such that } f(u) \cup \{q\} \in S_A \ \forall u \in x_0, \text{ and } f(u) \cup \{q\} \not\in S_A \ \forall u \in y_0. \text{ Define } g(p) = q. \]

\[ (\Leftarrow) \text{ Let } x, y \subseteq S_A \text{ be as in the hypothesis. Take } T = x \cup y, a = \cup T \text{ and } S_a \text{ the family generated by } T. \text{ Then } (a, S_a) \text{ is a finite polyhedron and } (a, S_a) < \mathcal{A}. \text{ Pick any } p \not\in a. \text{ Let } b = a \cup \{p\}, T_b = \{u \cup \{p\} : u \in x\} \cup y, \text{ and } S_b \text{ the family generated by } T_b. \text{ Then } (b, S_b) \text{ is a one-point extension of } (a, S_a); \text{ so there is an embedding } \]
\[ (b, S_b) \xrightarrow{g} \mathcal{A} \text{ satisfying } g \upharpoonright a = \text{id}. \text{ Take } q = g(p). \] □

5. The random polyhedron

The article of P. Erdos [2] is among the first approaches to the geometric properties of the random graph by means of probability methods. In this section we will study the universality property of countable random polyhedra. Here we will follow some arguments of [10], where the property is also studied for random polyhedra.

5.1. Definition of the random polyhedron. Hold a coin and assume that the probability of getting heads is \( p = \frac{1}{2} \). Define a family \( T \subseteq \mathbb{N}^{<\infty} \) as follows: For every \( u \in \mathbb{N}^{<\infty} \) flip the coin, and say that \( u \in T \) if and only if you get heads. Set
\[ S := \{v : (\exists u \in T) \ v \subseteq u\} \]
and set \( \omega := \bigcup S \). Let us write, from now on, \( S_\omega = S \) and \( T_\omega = T \). So \( (\omega, S_\omega) \) is the amalgamation of the random family of simplexes \( T_\omega \).

Theorem 5.1.1. Consider \( (\omega, S_\omega) \) as defined above. With probability 1, for each pair of finite, disjoint, non-empty subsets \( x, y \subseteq S_\omega \) satisfying that the elements of \( x \cup y \) are not comparable, there exists some \( q \in \omega \setminus (x \cup y) \) such that \( u \cup \{q\} \in S_\omega \) for all \( u \in x \), and \( u \cup \{q\} \not\in S_\omega \) for all \( u \in y \).

Proof: In \( 2^\omega \) take the product-metric topology, the σ-algebra of the Borel sets \( \mathcal{B} \) and the outer probability measure \( P \) which extends, by the Carathéodory method [4], the probability of finite coin flips. Given a finite subset \( z \subseteq \mathbb{N} \) and a finite tuple \( a \in 2^z \); the probability of the basic open set \( [a] = \{ \chi \in 2^\omega : \chi \upharpoonright z = a\} \) is \( P([a]) = \frac{1}{2^{|z|}}. \) Let \( X \xrightarrow{\chi} 2 \) be any random process defining the family
\[ T_\omega = X^{-1}(\{1\}) \] which generates \((\omega, S_\omega)\). Fix a bijection \(\omega^{[<\infty]} \overset{\phi}{\rightarrow} \mathbb{N} \). Then \(\chi = X\phi^{-1}\) is a random point in the probability space \((2^\mathbb{N}, \mathcal{B}, P)\). For any countable subset \(A \subset 2^\mathbb{N}\), we have \(P(A) = 0\). So, with probability 1, in a random countable flip one gets an infinite number of heads (see for instance \([3]\)). Since \(T_\omega\) is infinite with probability 1, so is \(\omega\).

Fix \(x, y \subset S_\omega\) as in the hypothesis and let \(n = |x \cup y|\). For each \(q \in \omega \cup (x \cup y)\), consider the proposition \(\varphi(q) := [(\forall u \in x) \ u \cup \{q\} \in S_\omega] \land [(\forall u \in y) \ u \cup \{q\} \not\in S_\omega]\). Let \(z_q = \{\phi(u \cup \{q\}) : u \in x \cup y\}\) and notice that the sets \(z_q\), with \(q\) ranging in \(\omega \cup (x \cup y)\), are pairwise disjoint. Define

\[ a_q : z_q \rightarrow 2 \quad a_q(\phi(u \cup \{q\})) = \begin{cases} 1 & u \in x \\ 0 & u \in y \end{cases} \]

Then the proposition \(\varphi(q)\) is equivalent to the statement \(\chi \upharpoonright z_q = a_q\). The probability that \(\varphi(q)\) holds is \(P(\chi \upharpoonright z_q = a_q) = \frac{1}{2^n}\). Therefore, by the definition of \(P\), for any finite subset \(F \subset [\omega \cup (x \cup y)]\),

\[ P(\forall q \in F \, \neg \varphi(q)) = \left(1 - \frac{1}{2^n}\right)^{|F|} \]

So

\[ P(\forall q \in \omega \cup (x \cup y) \, \neg \varphi(q)) = \prod_{q \not\in (x \cup y)} P(\chi \upharpoonright z_q \not= a_q) \leq \liminf_{F \subset [\omega \cup (x \cup y)]} P(\forall q \in F \, \neg \varphi(q)) = \liminf_{m \in \mathbb{N}} \left(1 - \frac{1}{2^n}\right)^m = \lim_{m \to \infty} \left(1 - \frac{1}{2^n}\right)^m = 0. \]

Therefore, \(\exists q \in \omega \cup (x \cup y) \, \varphi(q)\) holds with probability 1. This completes the proof. \(\square\)

**Corollary 5.1.2.** With probability 1, the following hold:

1. All infinite countable random polyhedra are ultrahomogeneous and isomorphic as countable structures.
2. \((\omega, S_\omega)\) contains finite simplexes of arbitrarily high dimension (cardinal).
3. Each finite polyhedron is rigidly embeddable on \((\omega, S_\omega)\).

**Proof:** (1) By Theorem 4.1.1, Proposition 4.4.2 and Theorem 5.1.1. (2) Take \(y = \emptyset\), use Theorem 5.1.1 and apply induction on \(|x|\). (3) By Part (2) and Lemma 3.3.1. \(\square\)

Part (1) of Corollary 5.1.2 allows us to call \((\omega, S_\omega)\) the infinite countable random polyhedron. We close this Section with the following.
Theorem 5.1.3. With probability 1, \( P = \text{Flim}(\mathcal{K}) \) is an infinite ordered polyhedron which is isomorphic to \((\omega, S_\omega)\), as a polyhedron, and to \((Q, \leq)\), as an ordered set.

Proof: By Corollary 5.1.2, \((\omega, S_\omega)\) is ultrahomogeneous and its age is \(\mathcal{K}\). By Theorem 4.1.1, \(P \cong (\omega, S_\omega)\) as a polyhedron. The second isomorphism is proved in a similar way, using the class of all finite linear orders. □

6. Topological Ramsey subspaces of \(\mathcal{P}\)

In this section, for every integer \(k > 0\), we will define a topological Ramsey space \(\mathcal{P}(k)\). It turns out that each \(\mathcal{P}(k)\) will be a closed subspace of \(\mathcal{P}\). In particular, \(\mathcal{P}(1) = \mathcal{E}\), Ellentuck’s space; and \(\mathcal{P}(2)\) is a topological Ramsey space whose elements are essentially the countably infinite ordered graphs. The corresponding set of approximations \(\mathcal{A}(2)\) is such that its closure under isomorphisms is essentially the class of finite ordered graphs, which is a Ramsey class (see [8]) whose Fraïssé limit is the ordered random graph. It is well-known that the automorphism group of the ordered random graph is, as in the case of the ordered random polyhedron, extremely amenable (see [6]).

6.1. The subspace \(\mathcal{P}(k)\).

Given \(k > 0\), consider pairs of the form \((A,S_A)\) where:

- \(A \in \mathbb{N}^{[\infty]}\).
- \(S_A \subseteq A^{[\leq k]}\).
- \(\bigcup S_A = A\).
- \(S_A\) is hereditary, i.e., \((u \subseteq v \& v \in S_A \Rightarrow u \in S_A)\).

Let us define \(\mathcal{P}(k)\) as the collection of all the pairs \((A,S_A)\) as above. Consider the restrictions to \(\mathbb{N} \times \mathcal{P}(k)\) of the approximation function \(r: \mathbb{N} \times \mathcal{P} \rightarrow \mathcal{A}\) and the restrictions to \(\mathcal{P}(k)\) and \(\mathcal{A}(k)\) of the pre-orders \(\leq\) and \(\leq_{\text{fin}}\) defined on \(\mathcal{P}\) and \(\mathcal{A}\).

Theorem 6.1.1. For every integer \(k > 0\), the triplet \((\mathcal{P}(k), r, \leq)\) is a topological Ramsey space. In fact, it is a closed subspace of \((\mathcal{P}, r, \leq)\).

Proof: Given \(k > 0\), to show that \((\mathcal{P}(k), r, \leq)\) is a topological Ramsey space, proceed as in the proof of Theorem 3.2.1. To show that is a closed subspace of \((\mathcal{P}, r, \leq)\), proceed as in the proof of Corollary 3.2.8. □

Actually, it is easy to show that given integers \(k' > k > 0\), \(\mathcal{P}(k)\) is a closed subspace of \(\mathcal{P}(k')\). As mentioned above, \(\mathcal{P}(1)\) is Ellentuck’s space \(\mathcal{E}\).

Now, let \(\mathcal{K}(k)\) denote the closure of \(\mathcal{A}(k)\) under isomorphisms. Proceeding as in the proofs of Theorems 4.3.2 and 4.3.3, we obtain the following:

Theorem 6.1.2. For every \(k > 0\), the class \(\mathcal{K}(k)\) of all finite ordered \(k\)-polyhedra is Ramsey. Furthermore, \(\mathcal{K}(k)\) is the age of some Fraïssé structure.

Corollary 6.1.3. For every \(k > 0\), the automorphism group of \(\text{Flim}(\mathcal{K}(k))\) is extremely amenable.

6.2. The random \(k\)-polyhedron.

For every \(k > 0\) probabilistically define \((\omega, S_\omega^k)\) with \(S_\omega^k \subseteq \omega^{[\leq k]}\), proceeding just as in Section 5 (for defining \((\omega, S_\omega)\)). Notice that the corresponding versions of Theorem 5.1.3 and Corollary 5.1.2 now can be easily proved in this context. It turns out that the resulting pair \((\omega, S_\omega^k)\) is characterized by the following, up to isomorphism: with probability 1, \((\omega, S_\omega^k)\) is isomorphic as
a polyhedron to $\text{FLim}(\mathcal{K}\mathcal{P}(k))$. We then call $(\omega, S^k_\omega)$ the random $k$-polyhedron. The following is the corresponding version of Theorem 5.1.3:

**Theorem 6.2.1.** Let $\mathbb{P}(k) = \text{FLim}(\mathcal{K}\mathcal{P}(k))$, the Fraïssé limit of $\mathcal{K}\mathcal{P}(k)$. Then, with probability 1, $\mathbb{P}(k)$ is isomorphic to $(\omega, S^k_\omega)$, as countable structure; and to $(\mathbb{Q}, \leq)$, as an ordered set.

So $\mathbb{P}_k$ is the ordered random $k$-polyhedron. It is also clear from the definitions that the random polyhedron contains an isomorphic copy of the random $k$-polyhedron.

### 6.3. The ordered random graph

The case $k = 2$ is of special notice. Observe that $\mathcal{K}\mathcal{P}(2)$ is essentially the class of finite ordered graphs. Hence, as it is well-known, its Fraïssé limit $\mathbb{P}_2 = \text{FLim}(\mathcal{K}\mathcal{P}(2))$ is the random ordered graph.

### 7. Universal minimal flows

Recall that every topological group $G$ has a universal minimal $G$-flow, unique up to isomorphism, which can be homomorphically mapped onto any other minimal $G$-flow. In this short Section we follow [6] in order to calculate the universal minimal $G$-flow, when $G$ is the automorphism group of the random polyhedron or the automorphism group of the random $k$-polyhedron, $k \in \mathbb{N} \setminus \{0\}$.

**Definition 7.0.1.** Let $L$ be a signature with $\{<\} \subseteq L$, and put $L_0 = L \setminus \{<\}$. Let $\mathcal{K}$ be a class of $L$-structures and put $K_0 = \mathcal{K} \upharpoonright L_0$. We say that $\mathcal{K}$ is reasonable if for every $A_0, B_0 \in K_0$, every embedding $\pi : A_0 \rightarrow B_0$, and every linear ordering $<$ on $A_0$ such that $A = < A_0, < > \in \mathcal{K}$, there is a linear ordering $<'$ on $B_0$, so that $B = < B_0, < > \in \mathcal{K}$ and $\pi : A \rightarrow B$ is also an embedding. Also, we say that $\mathcal{K}$ satisfies the ordering property if for every $A_0 \in K_0$ there exists $B_0 \in K_0$ such that for every linear ordering $<$ on $A_0$ and every linear ordering $<'$ on $B_0$, if $A = < A_0, < > \in K$ and $B = < B_0, < > \in K$ then $\mathcal{A} \leq \mathcal{B}$.

The class $\mathcal{K}\mathcal{P}_0$ is the reduct of $\mathcal{K}\mathcal{P}$; i.e. $\mathcal{K}\mathcal{P}_0$ is the class of structures $\mathcal{A}_0$ obtained from structures $\mathcal{A} \in \mathcal{K}\mathcal{P}$ by dropping the symbol $<^\mathcal{A}$. In a similar way, for every $k \in \mathbb{N} \setminus \{0\}$ define the reduct $\mathcal{K}\mathcal{P}_0(k)$ of $\mathcal{K}\mathcal{P}(k)$. The classes $\mathcal{K}\mathcal{P}_0$ and $\mathcal{K}\mathcal{P}_0(k)$ are ages of Fraïssé structures. The non ordered random polyhedron $(\omega, S_\omega)$ is the Fraïssé limit $\mathbb{P}_0$ of the class $\mathcal{K}\mathcal{P}_0$, and the Fraïssé limit $\mathbb{P}_0(k)$ of the class $\mathcal{K}\mathcal{P}_0(k)$ is the non ordered random $k$-polyhedron $(\omega, S^k_\omega)$. In fact, if $L \supseteq \{\leq\}$ is the signature of the ordered polyhedra and $L_0 = L \setminus \{\leq\}$ then, in the terminology of [6], $\text{FLim}(\mathcal{K}\mathcal{P}) \upharpoonright L_0 = \text{FLim}(\mathcal{K}\mathcal{P}_0)$ and $\text{FLim}(\mathcal{K}\mathcal{P}(k)) \upharpoonright L_0 = \text{FLim}(\mathcal{K}\mathcal{P}_0(k))$. These facts and Proposition 5.2 of [6] imply the following:

**Lemma 7.0.2.** The classes $\mathcal{K}\mathcal{P}$ and $\mathcal{K}\mathcal{P}(k)$, $k \in \mathbb{N} \setminus \{0\}$, are reasonable. □

Now, following [6] again, in order to calculate the universal minimal flows for the groups $\text{Aut}(\mathbb{P}_0)$ and $\text{Aut}(\mathbb{P}_0(k))$ we need to show the following:

**Lemma 7.0.3.** The classes $\mathcal{K}\mathcal{P}$ and $\mathcal{K}\mathcal{P}(k)$, $k \in \mathbb{N} \setminus \{0\}$, satisfy the ordering property.

**Proof:** We will prove that $\mathcal{K}\mathcal{P}$ satisfies the ordering property. The rest can be done in an analogous way. We will proceed as in the proof of Theorem 2 of [7], where it is proven that the class of all the finite ordered graphs satisfies the ordering property. So, fix $A_0 = (a, S_0) \in \mathcal{K}\mathcal{P}_0$ and a linear ordering $(a \leq)$. We will show that there exists $B_0 = (b, S_0) \in \mathcal{K}\mathcal{P}_0$ such that for every ordering $(b, \leq)$ there exists
a monotone mapping \( f : (a, \leq) \to (b, \leq) \) which is an embedding \((a, S_a) \to (b, S_b)\).

Let \( m \) be the number of finite polyhedra with set of vertices \( a \) which are isomorphic to \((a, S_a)\). As is the case of finite graphs, if \( m = 1 \) then we can take \((b, S_b) = (a, S_a)\), and therefore either \( S_a = \emptyset \) or \( S_a = a^{[\leq]} \), for some \( k \leq |a| \). So suppose \( m > 1 \).

By the Lemma in page 418 of [7], for some large enough integer \( n \), there exists a finite polyhedron \((c, S_c)\) such that its set of maximal faces \( T_c \) satisfies \(|T_c| = n^2\) and \((\forall u \in T_c) |u| = |a|\).

Let \( F_c \) be the class of all finite polyhedra \((c, \hat{S}_c)\), with \( c \) as set of vertices, such that:

1. For every \( u \in T_c \), \((u, \hat{S}_c \upharpoonright u) \simeq (a, S_a)\).
2. \( \hat{S}_c = \bigcup_{u \in T_c} \hat{S}_c \upharpoonright u \).

Notice that \(|F_c| = m^{n^2}\). On the other hand, for every linear ordering \( \leq \) on \( c \), there are less than \((m - 1)^{n^2} + 1 \) elements \((c, \hat{S}_c)\) of \( F_c \) for which there is no monotone embedding

\[(a, \leq), S_a \to ((c, \leq), \hat{S}_c)).\]

Therefore, the cardinality of set of those \((c, \hat{S}_c)\) \( \in F_c \) admitting an ordering \((c, \leq)\) for which there exists no embedding \(((a, \leq), S_a) \to ((c, \leq), \hat{S}_c))\) is less than \(m!(m-1)^{n^2}\), which is \(o(m^n)\). This completes the proof. \( \square \)

Given a signature \( L \) with \( \{<\} \subseteq L \) and \( L_0 = L \setminus \{<\} \), let \( \mathcal{K} \) be a reasonable class of \( L \)-structures and \( \mathcal{F} = F\text{Lim}(\mathcal{K}) \). Let \( F \) be the universe of \( \mathcal{F} \) and \( F_0 = \mathcal{F} \upharpoonright L_0 \). A linear ordering \( < \) on \( F \) is \( \mathcal{K} \)-admissible if for every finite substructure \( A_0 \leq F_0 \), we have \( A = < A_0, < [ A_0 > \in \mathcal{K} \), where \( A_0 \) is the universe of \( A_0 \). By Lemma 7.0.3, in virtue of Theorems 4.3.2 and 6.1.2 above, and Theorem 7.5(ii) of [6] we obtain the following:

**Theorem 7.0.4.** Let \( \mathcal{P}_0 = F\text{Lim}(\mathcal{K}^\omega_0) = (\omega, S_\omega) \) be the random polyhedron and let \( \mathcal{P}_0(k) = F\text{Lim}(\mathcal{K}^\omega_0(k)) = (\omega, S_\xi^k) \), \( k \in \mathbb{N} \setminus \{0\} \), be the random \( k \)-polyhedron.

The following holds:

1. The universal minimal \( \text{Aut}(\mathcal{P}_0) \)-flow is the metrizable \( \text{Aut}(\mathcal{P}_0) \)-flow of \( \mathcal{K} \)-admissible orderings on \( \omega \).

2. The universal minimal \( \text{Aut}(\mathcal{P}_0(k)) \)-flow is the metrizable \( \text{Aut}(\mathcal{P}_0(k)) \)-flow of \( \mathcal{K} \)-admissible orderings on \( \omega \).

**8. Final Comment**

The phenomena studied in this article reveal that, in general, there seems to be a tight relationship between a family of topological Ramsey spaces, and Ramsey classes of finite structures, extremely amenable automorphism groups, universal minimal flows, etc. This raises several questions. For instance, consider the abstract setting introduced in [11]. Given a Ramsey class \( \mathcal{K} \) of (ordered) structures, what is the precise description of a topological Ramsey space \( \mathcal{R} \) (if any), such that \( \mathcal{K} \) is the closure of \( \mathcal{AR} \)? Is it possible to characterize the family of topological Ramsey spaces \( \mathcal{R} \) for which the class \( \mathcal{AR} \) generates a Ramsey class of structures? On the other hand, a deeper study of the random polyhedron in itself and in relation to the random graph, from a wide point of view including approaches from model theory, graph theory, combinatorics, topology, dynamics and Ramsey theory is needed. We believe this work is a step in that direction.
References