L E C T U R E  1 3

Linear Programming Duality

1 Introduction

Consider the following linear program LP₁ in standard form:

Maximize $10x₁ + 6x₂$

Subject to

$$x₁ + 5x₂ ≤ 7 \quad (I)$$
$$-x₁ + 2x₂ ≤ 1 \quad (II)$$
$$3x₁ - x₂ ≤ 5 \quad (III)$$

$x₁, x₂ ≥ 0$

Let $z^*$ denote the optimum value of the objective function that satisfies the above linear program. What is an upper bound $z^*$? Multiplying (I) by 10 and comparing the left side with the objective function, we notice that 70 would be an upper bound, given that $x₁, x₂$ are nonnegative.

Can we improve 70? Consider $3*(I+II)$. This would give us 36. How about $I+2*III$? This is $7x₁ + 3x₂ ≤ 17$. Multiplying both sides by 2, and comparing with the objective function, we notice again that 34 is a better upper bound.

Instead of trying to find a linear combination of the constraints in an ad hoc way, let us formulate this process as another optimization problem. Denote the multiplier of I, II, and III by $y₁$, $y₂$, and $y₃$ respectively. In order that we compare the resulting sum of the constraints against the objective function, we require that these multipliers be nonnegative.

$y₁, y₂, y₃ ≥ 0$

Since, we want the best upper bound, we should minimize the expression obtained by the right-hand sides by the respective multipliers and adding them. That is,

Minimize $7y₁ + y₂ + 5y₂$

The corresponding sum from the left-hand side is
Rearranging this expression,

\[(y_1 - y_2 + 3y_3) x_1 + (5y_1 + 2y_2 - 3y_3) x_2\]

Comparing the coefficient of \(x_1\) and \(x_2\) with those of the objective function, and noting that this is an upper bound on \(z^*\), we conclude that

\[\begin{align*}
y_1 - y_2 + 3y_3 &\geq 10 \\
5y_1 + 2y_2 - 3y_3 &\geq 6
\end{align*}\]

In summary, we have

Minimize \(7y_1 + y_2 + 5y_2\)

Subject to

\[\begin{align*}
y_1 - y_2 + 3y_3 &\geq 10 \\
5y_1 + 2y_2 - 3y_3 &\geq 6
\end{align*}\]

\[y_1, y_2, y_3 \geq 0\]

Another linear program. The one we just derived is called the dual of the original (primal) program.

In the next section, we give an algebraic method to construct a dual for a given primal linear program.

## 2 Dual Linear Program

For a given linear program in the standard form,

\[
\begin{align*}
\text{maximize } & \mathbf{c}^T \mathbf{x}, \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0 \\
\text{where } & A \text{ is mxn matrix, } \mathbf{x} \text{ is an nx1 solution vector, } \mathbf{c} \text{ is nx1 vector, and } \mathbf{b} \text{ is an mx1 vector, it's dual is } \end{align*}
\]

\[
\begin{align*}
\text{minimize } & \mathbf{b}^T \mathbf{y}, \text{ subject to } A^T \mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq 0
\end{align*}
\]

where \(\mathbf{y}\) is an nx1 solution vector.

**(Weak Duality Theorem)** If \(\overline{\mathbf{x}}\) and \(\overline{\mathbf{y}}\) are feasible solutions to (IV) and (V) respectively, then \(c^T \overline{\mathbf{x}} \leq b^T \overline{\mathbf{y}}\).

Proof: Refer to the book (pp. 806).
3 Dual Linear Program With Mixed Inequalities

Consider the linear program LP₁ with \( \geq \) replaced by \( = \) in (III). That is, change (III) to

\[
3x_1 - x_2 = 5
\]

To convert this to the standard form, we replace this by

\[
3x_1 - x_2 \leq 5
\]

\[
3x_1 - x_2 \geq 5
\]

To remove \( \geq \), we multiply the last inequality by \(-1\) resulting in

\[
-3x_1 + x_2 \leq -5
\]

To summarize, we have the following linear program LP₁':

Maximize \( 10x_1 + 6x_2 \)

Subject to

\[
\begin{align*}
x_1 + 5x_2 & \leq 7 \quad \text{(VI)} \\
-x_1 + 2x_2 & \leq 1 \quad \text{(VII)} \\
3x_1 - x_2 & \leq 5 \quad \text{(VIII)} \\
-3x_1 + x_2 & \leq -5 \quad \text{(IX)}
\end{align*}
\]

\( x_1, x_2 \geq 0 \)

Let us find the dual for LP₁'. Denote the multipliers for the 4 constraints by \( y_1, y_2, y_3', y_3'' \). The dual is

Minimize \( 7y_1 + y_2 + 5y_3' - 5y_3'' \) or \( 7y_1 + y_2 + 5(y_3' - y_3'') \)

Subject to

\[
\begin{align*}
y_1 - y_2 + 3y_3' - 3y_3'' & \geq 10 \quad \text{(X)} \\
5y_1 + 2y_2 - y_3' + y_3'' & \geq 6 \quad \text{(XI)}
\end{align*}
\]

\( y_1, y_2, y_3', y_3'' \geq 0 \)

or

Minimize \( 7y_1 + y_2 + 5(y_3' - y_3'') \)

Subject to

\[
\begin{align*}
y_1 - y_2 + 3(y_3' - y_3'') & \geq 10 \quad \text{(X)} \\
5y_1 + 2y_2 - (y_3' - y_3'') & \geq 6 \quad \text{(XI)}
\end{align*}
\]

\( y_1, y_2, y_3', y_3'' \geq 0 \)
Replacing \((y_3' - y_3'')\) with \(y_3\) and getting rid of the nonnegativity constraints on \(y_3'\) and \(y_3''\), we get

\[
\text{Minimize } 7y_1 + y_2 + 5y_3
\]

Subject to

\[
y_1 - y_2 + 3y_3 \geq 10 \quad \text{(X)}
\]
\[
5y_1 + 2y_2 - y_3 \geq 6 \quad \text{(XI)}
\]
\[
y_1, y_2 \geq 0
\]

In general, if the \(i\)th constraint is an equality (=) in a primal LP, then the \(i\)th variable would not have the nonnegativity constraint in its dual and vice versa.

### 4 Reading Off a Solution to Dual from SIMPLEX

If the last slack form for the primal is

\[
z = \nu' + \sum_{j \in N} c_j' x_j
\]
\[
x_i = b_i' - \sum_{j \in N} a_{ij}' x_j \text{ for } i \in B
\]

where \(N\) and \(B\) are the sets of nonbasic and basic variables.

The optimal dual solution is

\[
y_i' = -c_{n+i}' \text{ if } (n+i) \in N, \text{ and } 0 \text{ otherwise}
\]

Refer to the book (pp. 806-807) for more details.