Using derivatives to find the shape of a graph

Example 1
The graph of \( y = x^2 \)

is decreasing for \( x < 0 \) and increasing for \( x > 0 \). Notice that where the graph is decreasing the slope of the tangent line, and therefore the derivative, is negative, and where the graph is increasing the slope and derivative are positive.

Example 2
Now consider the graph of \( y = 2x^3 - 3x^2 - 36x + 17 \).
You can see that this has a local maximum and a local minimum. We can find the exact coordinates by differentiating: \( y' = 6x^2 - 6x - 36 \). If we set this to zero and solve: \( 6x^2 - 6x - 36 = 0 \), then the solution is: \( \{ x = -2 \}, \{ x = 3 \} \). Looking at the graph you can see that the local maximum is at \( x = -2 \) and the local minimum is at \( x = 3 \). Now look at where the slope is positive or negative

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of slope</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -2))</td>
<td>positive</td>
<td>at ( x = -3, \ y' = 18 )</td>
</tr>
<tr>
<td>((-2, 3))</td>
<td>negative</td>
<td>at ( x = 0, \ y' = -36 )</td>
</tr>
<tr>
<td>((3, \infty))</td>
<td>positive</td>
<td>at ( x = 4, \ y' = 36 )</td>
</tr>
</tbody>
</table>

This pattern holds in general:

**Definition**

A function is *increasing* on an interval \( I \) if \( x_1 < x_2 \) implies \( f(x_1) < f(x_2) \)

A function is *decreasing* on an interval \( I \) if \( x_1 < x_2 \) implies \( f(x_1) > f(x_2) \)

**Theorem**

If \( f'(x) > 0 \) for all \( x \) in an interval \( I \), then \( f \) is increasing on \( I \).

If \( f'(x) < 0 \) for all \( x \) in an interval \( I \), then \( f \) is decreasing on \( I \).

Looking at these two examples you can see that at a local extrema the slope of the graph changes from positive to negative. This gives us the following test.

**First Derivative Test**

If \( x = c \) is a critical point of \( f \), then

1. \( f \) has a local maximum if \( f' \) changes sign from positive to negative at \( x = c \).
2. \( f \) has a local minimum if \( f' \) changes sign from negative to positive at \( x = c \).
3. \( f \) has no local extreme if \( f' \) has the same sign on both sides of \( c \).
4. \( f \) has both a local maximum and a local minimum at \( x = c \) if the derivative is zero on an open interval containing \( c \).

**Exercise**

Check this test for the previous two examples.
Example 3

Let \( f(x) = x^5 - 5x^4 + 5x^3 - 15 \)

It is clear that there is a local minimum between 2 and 4. Let’s zoom in on the interval \((-0.5, 1.5):\)

You can see that there is a local maximum around \( x = 1, \) and something funny is happening at \( x = 0. \)

First let’s find the critical points. The derivative is \( f'(x) = 5x^4 - 20x^3 + 15x^2. \) Setting this to zero and solving gives \( 5x^4 - 20x^3 + 15x^2 = 0, \) with solution \{ \( x = 0 \), \( x = 0 \), \( x = 1 \), \( x = 3 \). \) So the local maximum is at \( x = 1, \) and the local minimum is at \( x = 3. \) What happens at \( x = 0? \) If we put in values of \( x \) close to zero into \( f'(x), \) then we find that the derivative is positive on both sides of \( x = 0, \) so there is no local extremum there.
We have said that if a function is increasing, then its derivative is positive, and if it is decreasing, then the
derivative is negative. We can apply this to the first derivative test. If the derivative goes from positive to
negative, then it is decreasing, and will have negative slope. if the derivative goes from negative to positive,
then it is increasing and will have positive slope. This gives a better test for local extrema:

**Second Derivative Test**
Suppose \( f \) is a differentiable function and \( f'(c) = 0 \).

- If \( f''(c) < 0 \), then \( f \) has a local maximum at \( x = c \).
- If \( f''(c) > 0 \), then \( f \) has a local minimum at \( x = c \).

**Concavity**
Look at the graph of \( y = x^3 \)

On the right of the \( y \)-axis we say the graph is **concave up**, while on the left we say the graph is **concave down**. If you calculate the first derivative, then you can see that the graph is concave up when the derivative
is increasing, and concave down when the derivative is decreasing:

**Definition**
- The graph of \( y = f(x) \) is **concave up** when \( y' \) is increasing, or in other words, \( y'' > 0 \)
- The graph of \( y = f(x) \) is **concave down** when \( y' \) is decreasing, or in other words, \( y'' < 0 \)
- A point on the graph of \( y = f(x) \) where there is a tangent line and the concavity changes is called a
  **point of inflection**.
Example 4
Find where the graph $y = -2x^3 + 6x^2 - 3$ is increasing, decreasing, concave up and down; find extrema and inflection points.

First differentiate twice:
$$y' = -6x^2 + 12x$$
$$y'' = -12x + 12$$

- **Extrema:** solve $y' = 0$:
  $$-6x^2 + 12x = 0$$
  $$6x(2 - x) = 0$$
  $$x = 0, 2$$
  This gives critical points at $x = 0$ and $x = 2$. Now apply the second derivative test. At $x = 0$, $y'' = 12$, so this is a local minimum. At $x = 2$, $y'' = -12$, so this is a local maximum.

- **Increasing/decreasing:** Look for where $y' > 0$, and $y' < 0$. We know that $y' = 0$ at $x = 0$ and $x = 2$ so it must be positive and negative elsewhere. Checking $y'$ at $x = -1, 1, 3$ shows that $y'$ is negative and hence the function is decreasing on $(-\infty, 0)$ and $(2, \infty)$, and $y'$ is positive and hence the function is increasing on $(0, 2)$.

- **Inflection points:** Since the graph has tangent lines everywhere we need only look for where the second derivative is zero, and changes sign.
  $$y'' = 0$$ when $-12x + 12 = 0$, i.e. at $x = 1$.
  At $x = 0$, $y'' = 12$ and at $x = 2$, $y'' = -12$. Thus the graph has an inflection point at $x = 1$.

- **Concave up and down:** from the previous part $y'' < 0$ and hence the graph is concave down when $x > 1$, and $y'' > 0$ and hence the graph is concave up when $x < 1$. 
