Math 3400: Geometry Final
Due Friday 16th November

Please do not collaborate on this final. Most of the material for problems 1–3 may be found in any basic geometry or linear algebra textbook. Feel free look it up, but you must follow the directions in the question.

1. The Cauchy—Schwarz inequality for $\mathbb{R}^n$ says that if $x, y \in \mathbb{R}^n$, then $|\langle x, y \rangle| \leq |x||y|$ where $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ for $x = (x_1, \ldots, x_n)$.

   a) Prove the Cauchy—Schwarz inequality for $x$ or $y$ equal to zero.

   b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) = |x+ty|^2$ for $t \in \mathbb{R}$. Clearly $f$ is always positive.
      
      (i) Under what condition is $f$ zero?

      (ii) Express $f$ as a polynomial in $t$.

      (iii) Under what condition on $a, b, c$ is the polynomial $at^2 + bt + c$ in $t$ always greater than or equal to zero?

      (iv) What does this mean for $f$?

   c) Prove the Cauchy—Schwarz inequality for $x$ and $y$ not equal to zero using the previous part.

   d) When is the Cauchy—Schwarz inequality an equation?

2. Use the Cauchy–Schwarz inequality to prove the triangle inequality for vectors in $\mathbb{R}^n$: $|x+y| \leq |x| + |y|$ ($x, y \in \mathbb{R}^n$).

3. Use problems 1. and 2. to prove that if $|x+y| = |x| + |y|$ ($x, y \in \mathbb{R}^n$), then one vector is a positive multiple of the other.

4. Prove the Polarization Identity:

\[
\langle x, y \rangle = \frac{1}{2}(|x|^2 + |y|^2 - |x-y|^2) \quad (x, y \in \mathbb{R}^n).
\]

5. Prove the following equation:

\[
|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2) \quad (x, y \in \mathbb{R}^n).
\]

   What is the geometric significance of this?

6. Prove the triangle inequality in $\mathbb{E}^n$: $d(P, R) \leq d(P, Q) + d(Q, R)$ ($P, Q, R \in \mathbb{E}^n$).
7. Prove that if \( \ell = P + [v] \) is a line in \( E^2 \) and \( Q \in \ell, Q \neq P \), then \( \ell = \{ tP + (1 - t)Q : t \in \mathbb{R} \} \).

8. The previous problem provides the motivation for the following definition: a point \( X \in E^2 \) is **between** distinct points \( P, Q \in E^2 \) if \( X = tP + (1 - t)Q \) for some \( 0 < t < 1 \).

   Prove that \( X \) lies between \( P \) and \( Q \) if and only if \( d(P,X) + d(X,Q) = d(P,Q) \) (use problem 3.).

9. Let \( T \) be an isometry of \( E^2 \) that fixes the origin (i.e. \( T(0,0) = (0,0) \)).

   a) Prove that if \( P, Q \in E^2 \) and \( 0 \leq t \leq 1 \), then \( T(tP + (1 - t)Q) = tTP + (1 - t)TQ \).

   b) By making appropriate choices of \( t, P \) and \( Q \) in part (i) show that \( T \) is linear.

   c) Use the linearity of \( T \) and the fact that

   \[
   x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in E^2 \text{ may be written as } x = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
   \]

   to show that \( Tx = Ax \ (x \in E^2) \) for some invertible \( 2 \times 2 \) matrix \( A \).

10. Let \( \rho \) be a non-trivial rotation in \( E^2 \) with centre \( P \) and let \( v \) be any vector in \( \mathbb{R}^2 \). Show that \( \tau_v \rho \) is a rotation and find its centre.

11. Let \( P, Q, R, S \) be distinct points in \( \mathbb{R}^2 \), no three of which are collinear. Let \( A, B, C, D \) be the mid-points of \( PQ, QR, RS, SP \) respectively. Prove that \( AB \) is parallel to \( CD \) and \( AD \) is parallel to \( BC \).

12. Let \( x = (1,0,0), y = (1,1,0), z = (1,0,1), w = (1,1,1) \) be homogeneous coordinate vectors for points in \( P^2 \). Let \( \ell \) be the line joining \( \pi x \) and \( \pi y \), and let \( m \) be the line joining \( \pi z \) and \( \pi w \). Find \( \ell \cap m \).

13. Let \( \ell \) and \( \ell' \) be distinct lines in \( P^2 \), and let \( C \in P^2 \) be a point not on either line. The **perspectivity** \( [C; \ell \rightarrow \ell'] \) is the mapping \( \alpha \) that sends each point \( P \in \ell \) to the intersection of the line \( \overline{PC} \) with \( \ell' \).

   a) Verify that the mapping \( \alpha \) is well-defined.

   b) Prove that \( \alpha \) is a bijection with exactly one fixed point.

   c) Prove that \( \alpha^{-1} \) is a perspectivity.

   d) Prove that the composition of two perspectivities need not be a perspectivity.

   e) Given four distinct points \( P, Q, P', Q' \), prove that there is a unique perspectivity taking \( P \) to \( P' \) and \( Q \) to \( Q' \).