Abstract. We discuss aspects of Euclidean geometry including isometries of the plane, affine
transformations in the plane and symmetry groups. We then explore similar concepts in the sphere
and projective space, and explore elliptic and hyperbolic geometry.

1. Introduction

This course is intended for graduate and upper level undergraduate students. Only a basic math
background is required: an introduction to proofs and a familiarity with vector and matrix algebra
(scalar product, cross product, inverses of matrices) in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Almost every area of mathematics may be viewed from a geometric viewpoint and this viewpoint
often leads to new results or new proofs of existing results. It is the intuition that one can gain
from looking at a problem geometrically that can be very important in finding a solution. However,
in order to best appreciate this approach one must first learn the basic ideas that underpin the
study of geometry.

Geometry as a subject in itself was first studied by listing the axioms that seemed to embody the
fundamental concepts and then using these axioms to prove “known” results, i.e. to put every-day
ideas into a rigorous framework. In Section 2 we briefly discuss this axiomatic approach and show
how it can be used to construct the familiar Euclidean geometry. In the following sections we use
a more analytic method to investigate the properties of the Euclidean plane, the 2-Sphere and the
projective plane.

2. Axiomatic Geometry

Each theorem in a deductive system must be derived from previous theorems which must in turn
be derived from previous theorems. Since this process must start somewhere we take a set of agreed
upon notions as fact. These are called axioms or postulates. From these we may deduce theorems.

We shall illustrate the axiomatic method with an example.

Definition 2.1. Blobs and rocks are objects that satisfy the following axioms:

- **P1** There exist at least two rocks.
- **P2** For any two distinct rocks there exists a unique blob containing both.
- **P3** Given any blob there is a rock not in it.
- **P4** Given any blob $B$ and a rock $r$ not in $B$ there exists a unique blob $C$ containing $r$ disjoint
  from $B$. 
In the spirit of being unable to define every concept we want to use, we have not defined what it means for one object to contain another. In fact “contains” is simply a binary relation, but does not necessarily mean “live inside”. Notice that there is no axiom above preventing rocks from containing blobs.

We need some more definitions before we can proceed.

**Definition 2.2.** If two blobs $B$ and $C$ contain the same rock $r$, then we say that $B$ and $C$ intersect or meet in $r$.

**Definition 2.3.** Two blobs $B$ and $C$ are disjoint if they have no rock in common.

**Definition 2.4.** The unique blob containing the pair of rocks $r$ and $s$ will be denoted by $rs$.

Note that if a blob contains rocks $p$, $q$ and $r$ then $pq$, $qr$ and $pr$ all denote the same blob. We are not claiming that blobs may contain 3 rocks; in fact this is one of the questions that we shall attempt to answer. We would like to know if there is a minimum or maximum for the number of rocks, the number of blobs, and the number of rocks a blob may contain.

**Theorem 2.5.** There are at least two disjoint blobs.

**Proof.**

(P1) Let $p$ and $q$ be two different rocks.
(P2) Let $B = pq$ be the blob containing $p$ and $q$.
(P3) Let $r$ be a rock not in $B$ and, consequently, different from $p$ and $q$.
(P4) Let $C$ be a blob disjoint from $B$ containing $r$.

It is now clear that $B$ and $C$ are two disjoint blobs as required. □

**Theorem 2.6.** Every blob contains at least one rock.

**Proof.** Assume that there exists an empty blob $B$. Since $B$ is empty it follows that any non-empty blob is disjoint from $B$. Let $r$ and $s$ be two distinct rocks (P1). Then $A = rs$ (P2) is disjoint from $B$. Let $t$ be a rock not in $A$ (P3) and $C = rt$ (P2). Blob $C$ is also disjoint from $B$ and $C \neq A$. We now have two different blobs, $A$ and $C$, disjoint from $B$ that contain $r$. This contradicts (P4) and hence there are no empty blobs. □

**Theorem 2.7.** Two distinct blobs, each disjoint from a third blob are disjoint from each other.

**Proof.** Let $A$ and $B$ be two distinct blobs, both disjoint from a blob $C$. Suppose $A$ and $B$ are not disjoint. Let $r$ be a rock common to both $A$ and $B$. We have produced two different blobs that contain the same rock not in $C$ and are disjoint from $C$. This contradicts (P4) and hence the assumption that $A$ and $B$ are not disjoint. □

**Theorem 2.8.** Every blob contains at least two rocks.

**Proof.** Suppose there exists a blob $B$ containing exactly one rock $r$. Let $s$ be a rock not in $B$ (P3), and $C = rs$ (P2). Let $t$ be a rock not in $C$ (hence, not in $B$) (P3). Let $A$ be a blob containing $t$ that is disjoint from $C$ (P4). $A$ must be disjoint from $B$ as the only rock of $B$ is a rock of $C$. Blobs
Definition 2.9. Two sets $A$ and $B$, finite or infinite, have the same size if there is a one-to-one correspondence between the elements of set $A$ and the elements of set $B$.

Theorem 2.10. The number of blobs containing a given rock $r$ is the same as the number of blobs containing any other rock $s$.

Proof. Let $r$ and $s$ be two different rocks (P1). We shall construct a bijection from the set of blobs containing $r$ and the set of blobs containing $s$. Let $B = rs$ be the unique blob containing $r$ and $s$ (P2). Let $t$ be a rock not in $B$ (P3) and $C$ a blob containing $t$ disjoint from $B$ (P4). Note that $B$ is the unique blob containing $r$ disjoint from $C$. Thus, each blob containing $r$, except $B$, will meet $C$ in a rock (P4) which is unique (P2). (Note that two different blobs cannot have two rocks in common). Thus there is a one-to-one correspondence, or bijection $\alpha$, from blobs through $r$ that meet $C$ to rocks in $C$. Repeating the above argument with $s$ we obtain a bijection $\beta$ from blobs through $s$ that meet $C$ to rocks in $C$. A one-to-one correspondence between blobs through $r$ and blobs through $s$ is now easily obtained by defining $f(B) = B$ and $f(A) = \beta^{-1}(\alpha(A))$, for blobs $A \neq B$ that contain $r$. □

Theorem 2.11. All blobs contain the same number of rocks, i.e., have the same size.

Proof. Let $B$ be a blob (P1,P2) and $r$ a rock not in $B$ (P3). Except for the unique blob containing $r$ disjoint from $B$ (P4), every blob containing $r$ has a unique rock in common with $B$ (P2, Theorem 2.7). Thus, different blobs containing $r$ determine different rocks of $B$ and each rock of $B$ determines a distinct blob containing $r$. Notice that the number of rocks in blob $B$ is one less than the number of blobs containing the rock $r$. Similarly, for any other blob $C$ let $s$ be a rock not in $C$. The number of rocks in $C$ is one less than the number of blobs containing $s$. Since the number of blobs containing $r$ is the same as the number of blobs containing $s$ (Theorem 2.10) it follows that the number of rocks in $B$ is the same as the number of rocks in $C$. □

Homework: Provide a model for the above axioms other than lines and points.

Solution 1. Consider a tetrahedron. Let rocks be the vertices of the tetrahedron and blobs be the edges. Notice how all the theorems proved above continue to hold.

Solution 2. Consider the same tetrahedron. Let rocks be the faces of the tetrahedron and blobs be the edges. Again, all the theorems proved above continue to hold.
3. Isometries in the Euclidean Plane

In this section we shall consider the Euclidean plane to be the vector space $\mathbb{R}^2$ together with the usual pythagorean distance. We then discuss functions from the plane to itself which preserve distance. These are called isometries. They are the familiar reflections, rotations, translations and glide reflections along with combinations of these. The goal of this section and the next is to provide a complete description of isometries.

3.1. Notation. We shall let $\mathbb{R}^n$ denote the vector space $\{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$. For $x, y \in \mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ is the inner or dot product of $x$ and $y$. We shall use this to define the standard distance function on $\mathbb{R}^n$ to be $d(P,Q) = |P - Q|$ where $|x| = \sqrt{\langle x, x \rangle}, x \in \mathbb{R}^n$. The $n$-dimensional Euclidean space $E^n$ may now be defined as $E^n = (\mathbb{R}^n, |\cdot|)$, i.e., the set $\mathbb{R}^n$ together with the distance function $d$.

3.2. Examples. This is not the only distance function that may be defined. Other spaces that are of interest in mathematics are the $\ell_p^n$ spaces ($1 \leq p \leq \infty$). For $1 \leq p < \infty$ define the function $|\cdot|_p : \mathbb{R}^n \to \mathbb{R}$ by

$$|x|_p = \left( \sum_{1}^{n} |x_i|^p \right)^{1/p} \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

and for $p = \infty$ let $|x|_\infty = \max_{1 \leq i \leq n} |x_i|$. We then define $\ell_p^n$ as $(\mathbb{R}^n, |\cdot|_p)$ for $1 \leq p \leq \infty$. Notice that $E^n = \ell_2^n$.

We can get an idea of how these different distance functions distort spacial relationships if we compare the “unit spheres” of these spaces, that is the locus of points $\{x \in \mathbb{R}^2, |x|_p = 1\}$. Shown below are the unit spheres for $\ell_p^2$, with $p = 1, 2$ and $\infty$ (working from the inside out).

\[\text{Definition 3.1.} \quad \text{An isometry of } E^2 \text{ is an onto function } T : E^2 \to E^2 \text{ such that } d(Tx, Ty) = d(x, y). \]

In other words, isometries preserve distance.

Isometries satisfy certain important properties:

- Isometries are one-to-one functions, since if $Tx = Ty$ then $0 = d(Tx, Ty) = d(x, y)$ which implies $x = y$.
- Isometries have inverses which are also isometries. Since an isometry $T$ is a bijection so is $T^{-1}$. Then $d(T^{-1}x, T^{-1}y) = d(TT^{-1}x, TT^{-1}y) = d(x, y)$.
The composition of two isometries is again an isometry. If $S$ and $T$ are isometries, then $d(STx, STy) = d(TxTy) = d(x, y)$.

It is clear that composition of isometries is associative and hence the set of isometries of $\mathbb{E}^2$ forms a group under composition. We denote this group by $I(2)$. (For the definition of a group see Appendix A.)

3.3. A Concrete Example. We make a small digression here to present a concrete example of some isometries which fix a given triangle using the familiar ideas of trigonometry.

Consider the points $A(0, 2), B(-\sqrt{3}, -1)$ and $C(\sqrt{3}, -1)$ in $\mathbb{E}^2$. We are interested in isometries $T : \mathbb{E}^2 \to \mathbb{E}^2$ that map the triangle $\triangle ABC$ to itself. For simplicity we denote this triangle by $\triangle$.

Suppose $S$ and $T$ are two such isometries that also satisfy $SA = TA$, $SB = TB$, and $SC = TC$. Must $S$ equal $T$? In other words, could there be a point $x$ in $\mathbb{E}^2$ such that $Tx \neq Sx$? The answer to this important question is no but we postpone the proof until later (see Theorem 3.2).

Since a vertex of $\triangle$ must map to a vertex of $\triangle$ and an isometry of our set is fully defined by its behavior on the vertices of $\triangle$, it follows that there are no more than $3! = 6$ permutations of a set of 3 elements. It is, in fact, easy to come up with 6 such isometries, which must then necessarily constitute all the isometries that map $\triangle$ to $\triangle$.

Let $[xyz]$ denote the unique isometry that satisfies $TA = x$, $TB = y$, $TC = z$. Our desired set of isometries includes the following (it would be useful to draw a picture as you are reading this list):

- $[ABC]$ the identity transformation, also denoted by $I$.
- $[CAB]$ a rotation by $2\pi/3$ around the origin.
- $[BCA]$ a rotation by $-2\pi/3$ around the origin.
- $[ACB]$ a reflection about the line through $A$ and the midpoint of $BC$.
- $[CBA]$ a reflection about the line through $B$ and the midpoint of $AC$.
- $[BAC]$ a reflection about the line through $C$ and the midpoint of $AB$.

Since the composition of two isometries results in an isometry we can compute a “multiplication table” under the composition operator $\circ$:

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<tr>
<th></th>
<th>$I$</th>
<th>$[CAB]$</th>
<th>$[BCA]$</th>
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<td>$[BAC]$</td>
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<td>$[CBA]$</td>
<td>$[CAB]$</td>
<td>$[BCA]$</td>
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</table>

As expected, the composition of two isometries yields an isometry and no new isometries are produced through this process. A quick glance through the above multiplication table for the isometries of $\triangle$ shows that this set, under composition, is a subgroup of $I(2)$.

The various isometries of $\triangle$ can be easily computed by studying their effect on the unit vectors $(1, 0)$ and $(0, 1)$. For example, $[CAB]$ exerts a rotation of $2\pi/3$ around the origin. Accordingly,
\[ CAB(1,0) = (\cos 2\pi/3, \sin 2\pi/3) = (-1/2, \sqrt{3}/2) \text{ and } CAB(0,1) = (-\sin 2\pi/3, \cos 2\pi/3) = (-\sqrt{3}/2, -1/2). \] This can be compactly represented in matrix form as:

\[
CAB(x,y) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -x - \sqrt{3}y \\ \sqrt{3}x - y \end{bmatrix}
\]

Similarly,

\[
BCA(x,y) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} x - \sqrt{3}y \\ \sqrt{3}x + y \end{bmatrix}
\]

\[
ACB(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}
\]

Consider now the line \( \ell_2 \) through \( B \) and the midpoint \( M = (\sqrt{3}/2, 1/2) \) of \( AC \). Since \( M = (\cos \pi/6, \sin \pi/6) \), we have that \( M \) makes an angle \( \pi/6 \) with the \( x \)-axis. Thus, when reflected about \( \ell_2 \), the vector \((1,0)\) becomes \((\cos 2\pi/6, \sin 2\pi/6) = (1/2, \sqrt{3}/2)\). Similarly, \((0,1)\) forms an angle \( \pi/3 \) with \( \ell_2 \). To reflect it we rotate it \( 2\pi/3 \) degrees obtaining \((\cos -\pi/6, \sin \pi/6) = (\sqrt{3}/2, -1/2)\).

Thus

\[
CBA(x,y) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x + \sqrt{3}y \\ \sqrt{3}x - y \end{bmatrix}
\]

Similarly,

\[
BAC(x,y) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x - \sqrt{3}y \\ -\sqrt{3}x - y \end{bmatrix}
\]

3.4. Reflection about an arbitrary line. We may directly calculate how to reflect a point \( x \) through an arbitrary line \( \ell \) that passes through a point \( p \). The reflected point is denoted by \( x' \). Let \( n \) be a unit normal to the line and \( \theta \), the angle between \( n \) and the vector \( x - p \) (see the diagram below). Thus, \( \langle x - p, n \rangle = |x-p| \cos \theta \) is the length of the projection of \( x-p \) onto a line orthogonal to \( \ell \), and \( (x-p,n) \) is the vector perpendicular to \( \ell \) from \( \ell \) to \( x \). Thus, to project \( x \) we must displace \( x \) twice this distance in a direction opposite to \( (x-p,n) \). In other words \( x' = x - 2\langle n, x-p \rangle n \).

\[
\begin{array}{c}
\ell \\
\quad \theta \\
\quad n \\
p \\
x' \quad x
\end{array}
\]

We now go back to the question of what it takes to uniquely define an isometry.

**Theorem 3.2.** If \( T \) is an isometry that maps a triangle \( \triangle ABC \) to itself, then \( T \) is unique, i.e., there is no other isometry \( S \) such that \( TA = SA \), \( TB = SB \), \( TC = SC \) and \( Tx \neq Sx \) for some \( x \in \mathbb{R}^2 \).
Proof. Assume $TA = SA = a$, $TB = SB = b$, $TC = SC = c$, where $a$, $b$, $c$ are not collinear. If $x \in \{A, B, C\}$ we are done. Otherwise, since both $T$ and $S$ are isometries,

$$d(TA, Tx) = d(A, x) = d(SA, Sx)$$

and hence $d(a, Tx) = d(a, Sx) = r_a > 0$ ,

$$d(TB, Tx) = d(B, x) = d(SB, Sx)$$

and hence $d(b, Tx) = d(b, Sx) = r_b > 0$ ,

$$d(TC, Tx) = d(C, x) = d(SC, Sx)$$

and hence $d(c, Tx) = d(c, Sx) = r_c > 0$ .

This means that both $Tx$ and $Sx$ must lie on the boundary of each of the circles centered at $a$, $b$, and $c$ with radii $r_a$, $r_b$ and $r_c$, respectively. The circles centered at $a$ and $b$ intersect at most twice. If they intersect once we are done. Otherwise, since $a$, $b$, $c$ are not collinear $c$ cannot be equidistant from both intersections and only one of them lies at distance $r_c$ from $c$. This intersection is $Tx = Sx$. □

Definition 3.3 (Subgroups of $I(2)$). We have the following interesting subgroups of $I(2)$ that will occur later:

$I(2)$ all isometries of $E^2$.

$O(2)$ isometries of $E^2$ that fix the origin, i.e., $O(2) = \{ T \in I(2), T(0,0) = (0,0) \}$. This is also called the orthogonal group of $E^2$.

$SO(2)$ all rotations about the origin.

Note that $SO(2)$ is a subgroup of $O(2)$ which is a subgroup of $I(2)$. $O(2)$ is, in a sense, “twice the size” of $SO(2)$ as we can obtain two members of $O(2)$ from each member of $T$ of $SO(2)$ by either applying a reflection or not before the rotation.

4. Reflections, Translations and Rotations in $E^2$

In this section we discuss reflections: isometries of $E^2$ given by reflecting in a line. The product of two reflections is either a translation or a rotation, and the product of three reflections is another reflection. In fact we shall see that any isometry may be written as the product of at most three reflections.

4.1. Reflections. In order to write down a formula for a reflection we first need to know how to reflect a point in a line.

Theorem 4.1. If $\ell$ and $m$ are perpendicular lines, then they have a unique point in common.

Proof. Let $\ell = P + [v]$ and $m = Q + [w]$ with $|v| = |w| = 1$. Since $\ell$ and $m$ are perpendicular it follows that $\langle v, w \rangle = 0$ and we may write

$$P - Q = \langle P - Q, v \rangle v + \langle P - Q, w \rangle w .$$

This gives $P - \langle P - Q, v \rangle v = Q + \langle P - Q, w \rangle w$ and setting $F = P - \langle P - Q, v \rangle v = Q + \langle P - Q, w \rangle w$ we see that $F$ is on both lines.

If there are two points $F, G$ on both $\ell$ and $m$, then $F - G \in [v] \cap [w] = \{0\}$ and hence $F - G = 0$, i.e. $F = G$. Thus the point $F$ is unique. □
The following corollary is an immediate consequence of this theorem.

**Corollary 4.2.** Let $X$ be a point and $\ell$ in $E^2$, then there exists a unique line $m$ through $X$ perpendicular to $\ell$ and

(i) $m = X + [n]$ where $n$ is a unit normal to $\ell$ (that is, if $[v]$ is the direction of $\ell$ with $v = (v_1, v_2)$, then $n = \pm(v_2/\sqrt{v_1^2 + v_2^2}, -v_1/\sqrt{v_1^2 + v_2^2})$);

(ii) $\ell$ and $m$ intersect in $F = X - (X - P, n) n$ where $P$ is any point on $\ell$;

(iii) $d(X, F) = |(X - P, n)|$.

We can now use the formula for the intersection of two perpendicular lines to find the equation for the reflection of a point in the line, and thus define reflections.

**Definition 4.3.** The reflection of a point $X$ in a line $\ell = P + [v]$ is the point $X'$ that satisfies

$$\frac{1}{2}(X + X') = F$$

where $F = X - (X - P, n) n$ is the intersection of $\ell$ with the line through $X$ perpendicular to $\ell$.

![Figure 1. The reflection of the point $X$ in the line $\ell$](image)

Now,

$$\frac{1}{2}(X + X') = F , \text{ so}$$

$$\frac{1}{2}X + \frac{1}{2}X' = X - (X - P, n) n \text{ and hence}$$

$$\frac{1}{2}X' = \frac{1}{2}X - (X - P, n) n \text{ thus}$$

$$X' = X - 2(X - P, n) n .$$

**Definition 4.4 (Reflection).** The **reflection in line $\ell$** (where $\ell = P + [v]$) is the bijection $\Omega_\ell : E^2 \to E^2$ given by

$$\Omega_\ell X = X - 2(X - P, n) n ,$$

where $n$ is a unit normal vector to $\ell$. 
4.2. **Translations.** We next investigate what happens when we compose two reflections. The result depends on whether or not the lines of reflection are parallel. In the following diagrams $X'$ is the reflection of $X$ in $\ell$, while $X''$ is the reflection of $X'$ in $m$.

![Figure 2. Reflection of $X$ in line $\ell$, then in line $m$](image)

If $m$ and $n$ are parallel lines, $P$ is on $m$ and $Q$ is at the foot of the perpendicular from $P$ to $n$, and $N$ is the unit normal to $m$ (and hence also to $n$), then

$$
\Omega_m \Omega_n X = \Omega_n X - 2(\Omega_n X - P, N)N
= X - 2(X - P, N)N + 2(X - Q, N)N
= X + 2(P - Q, N)N
= X + 2(P - Q)
$$

**Definition 4.5** (Translation). Let $\ell$ be a line and $m, n$ lines perpendicular to $\ell$. The transformation $\Omega_m \Omega_n$ is a **translation along** $\ell$. If $m \neq n$, then the translation is non-trivial.

**Theorem 4.6.** If $T$ is a non-trivial translation along a line $\ell$, then $\ell$ has a direction vector $v$ such that $Tx = x + v$ and conversely.

**Proof.** Let $N$ be a unit direction vector for $\ell$, let $P \in \mathbb{E}^2$ and let $\alpha, \beta$ be distinct lines perpendicular to $\ell$. Let $a, b$ be the unique real numbers such that $P + aN \in \alpha$ and $P + bN \in \beta$. Then

$$
\Omega_\alpha \Omega_\beta X = x + 2((P + aN) - (P + bN)) = x + 2(a - b)N
$$

If $T$ is not the identity, then $a \neq b$ and $2(a - b)N$ is the required direction.
Conversely, for any $\lambda \in \mathbb{R}$ let $T_\lambda x = x + \lambda N$ and let $a, b$ satisfy $\lambda = 2(a - b)$. Setting $\alpha = P + aN + [N^\perp]$ and $\beta = P + bN + [N^\perp]$, where $N^\perp$ is a unit normal vector to $\ell$, gives $T_\lambda = \Omega_\alpha \Omega_\beta$ as required.

**Definition 4.7.** For $v \in \mathbb{R}$ let $\tau_v$ be the translation $\tau_v x = x + v$.

### 4.3. Rotations.

Let $\ell = P + [v]$ be a line with $|v| = 1$. There exists a unique $\theta \in (-\pi, \pi]$ such that $v = (\cos \theta, \sin \theta)$. We shall use the unit normal to $v$, $N = (-\sin \theta, \cos \theta)$. We have shown that reflection in the line $\ell$ is given by

$$
\Omega_\ell X = X - 2\langle X - P, N \rangle N.
$$

Now let $\ell_0 = 0 + [v]$ be the line through the origin parallel to $\ell$; we shall show that $\Omega_\ell = \tau_P \Omega_{\ell_0} \tau_{-P}$.

We have

$$
\Omega_\ell X - P = X - P - 2\langle X - P, N \rangle N \quad \text{and}
$$

$$
\Omega_{\ell_0} X = X - 2\langle X, N \rangle N, \quad \text{hence}
$$

$$
\Omega_\ell X - P = \Omega_{\ell_0} (X - P).
$$

Thus $\Omega_\ell X = \Omega_{\ell_0} (X - P) + P$ as required.

Let $x = (x_1, x_2) \in \mathbb{R}^2$, then $\langle x, N \rangle = -x_1 \sin \theta + x_2 \cos \theta$ and writing the vectors in columns we have

$$
\Omega_{\ell_0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2(-x_1 \sin \theta + x_2 \cos \theta) \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} = 
\begin{bmatrix} (1 - 2 \sin^2 \theta)x_1 + (2 \sin \theta \cos \theta)x_2 \\ (2 \sin \theta \cos \theta)x_1 + (1 - 2 \cos^2 \theta)x_2 \end{bmatrix} =
\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$
The matrix
\[
\text{ref}_\theta = \begin{bmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{bmatrix}
\]
is therefore a reflection in the line through the origin with direction \((\cos \theta, \sin \theta)\).

We saw in Figure 2 that reflection in two non-parallel lines gives a rotation. If \(m\) is a second line through \(P\) with direction \((\cos \phi, \sin \phi)\), then
\[
\text{ref}_\theta \text{ref}_\phi = \begin{bmatrix}
\cos 2(\theta - \phi) & -\sin 2(\theta - \phi) \\
\sin 2(\theta - \phi) & \cos 2(\theta - \phi)
\end{bmatrix}.
\]
We write a matrix of this form as
\[
\text{rot}_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]
Since \(\text{rot}_\theta\) takes the vector \((1, 0)\) and \((0, 1)\) to \((\cos \theta, \sin \theta)\) and \((-\sin \theta, \cos \theta)\) respectively, we can think of \(\text{rot}_\theta\) as a rotation.

**Definition 4.8 (Rotation).** If \(\alpha\) and \(\beta\) are lines through a point \(P\), then the isometry \(\Omega_\alpha \Omega_\beta\) is called a rotation about \(P\).

### 4.4. Isometries that fix the origin.

Let \(T\) be an isometry such that \(T(0) = 0\). What is \(T\)?

Let us attempt to express \(T\) using matrices and vectors. Since \(T\) is affine, we have a matrix \(A\) and vector \(b\) with \(A\) invertible and \(Tx = Ax + b\).

\[
\begin{align*}
T(0) &= A(0) + b \\
T(0) &= 0 + b \\
T(0) &= b
\end{align*}
\]

Since \(T(0) = 0, b = 0\). So \(Tx = Ax, A\) invertible. Now we have to find \(A\). Let
\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
\(A\) is invertible, so \(ad - bc \neq 0\). We can also use the distance preserving property of isometries.

Consider the action of \(T\) on \((0, 0), (0, 1), (1, 0)\) and \((1, 1)\).

\[
\begin{align*}
T \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}. \quad \text{So } a^2 + c^2 = 1. \\
T \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}. \quad \text{So } b^2 + d^2 = 1. \\
T \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix}. \quad \text{So } (a + b)^2 + (c + d)^2 = 2.
\end{align*}
\]
\[
\Rightarrow a^2 + b^2 + c^2 + d^2 + 2ab + 2cd = 2 \\
\Rightarrow ab + cd = 0 \\
\Rightarrow cd = -ab
\]

Given \(Tx = Ax, T^{-1}x = A^{-1}x\).
\[ A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } \Delta = ad - bc. \]

\[ T^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{\Delta} \\ \frac{-c}{\Delta} \end{bmatrix} \]

\[ \Rightarrow \frac{d^2}{\Delta^2} + \frac{c^2}{\Delta^2} = 1 \]

\[ \Rightarrow c^2 + d^2 = \Delta^2. \]

\[ T^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{b}{\Delta} \\ \frac{a}{\Delta} \end{bmatrix} \]

\[ \Rightarrow \frac{b^2}{\Delta^2} + \frac{a^2}{\Delta^2} = 1 \]

\[ \Rightarrow a^2 + b^2 = \Delta^2. \]

\[ a^2 + b^2 + c^2 + d^2 = 2\Delta^2 \]

\[ \Rightarrow 2 = 2\Delta^2 \]

\[ \Rightarrow \Delta = \pm 1 \]

Let us reconstruct \( A \).

\[ a^2 + b^2 = a^2 + c^2 = b^2 + d^2 = c^2 + d^2 = 1; \quad cd = -ab. \]

Let \( a = \cos \theta; \ b = \pm \sin \theta. \) Since \( a^2 + c^2 = 1, \ c = \pm \sin \theta. \) Also \( c^2 + d^2 = 1 \implies d = \pm \cos \theta. \)

Lets pick each possibility.

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{bmatrix} = \text{ref}(\frac{\theta}{2})
\]

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} = \begin{bmatrix}
\cos(-\theta) & -\sin(-\theta) \\
\sin(-\theta) & \cos(-\theta)
\end{bmatrix} = \text{rot}(-\theta)
\]

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} = \text{rot}\theta
\]

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{bmatrix} = \begin{bmatrix}
\cos(-\theta) & \sin(-\theta) \\
\sin(-\theta) & -\cos(-\theta)
\end{bmatrix} = \text{ref}(\frac{-\theta}{2})
\]

Thus if \( T \) fixes the origin, it must be either a reflection or a rotation.
4.5. **Isometries that have a fixed point.** We have shown that if $T$ is an isometry with $T(0) = 0$, then $T$ is either a reflection or a rotation. If $T$ has a fixed point $P$, then $T$ is either a rotation about $P$ or a reflection in a line through $P$.

Since $T$ fixes $P$, $TP = P$. $\tau_px = x + P$. Then $\tau_-P\tau_p(0) = 0$.

$\tau_-P\tau_p(0) = \tau_-P(P) = \tau_-P(0) = 0$

Therefore $\tau_-P\tau_p = \text{ref}(\theta)$ or $\text{rot}(\theta)$. So $T = \tau_p\text{ref}(\theta)\tau_-P$ or $\tau_p\text{rot}(\theta)\tau_-P$.

4.6. **Isometries with no fixed points.** Let $T$ be an isometry with no fixed points. Let $P = T0$. Then $(\tau_-P)(0) = 0$.

So $\tau_-P = \text{ref}\theta$ or $\text{rot}\theta$. Hence $T = \tau_p\text{ref}\theta$ or $T = \tau_p\text{rot}\theta$.

We have thus proven the following theorems:

**Theorem 4.9** (Classification of isometries of $E^2$). An isometry of $E^2$ must be one of the following:

(i) $\text{ref}(\theta)$;
(ii) $\text{rot}(\theta)$;
(iii) $\tau_v\text{ref}(\theta)\tau_-v$;
(iv) $\tau_v\text{rot}(\theta)\tau_-v$;
(v) $\tau_v\text{ref}(\theta)$;
(vi) $\tau_v\text{rot}(\theta)$.

**Theorem 4.10.** The group $I(E^2)$ of isometries on $E^2$ is generated by \{ $\text{rot }\theta, \text{ref }\theta : \theta \in (-\pi, \pi]$ \} \cup \{ $\tau_P : P \in E^2$ \}.

**Theorem 4.11** (Three Reflection Theorem for parallel lines). If $\alpha$, $\beta$, $\gamma$ are all parallel, then there exists a line $\delta$ parallel to $\alpha$, $\beta$, $\gamma$ such that $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta$.

**Proof.** Let $\ell$ be a line perpendicular to $\alpha$, $\beta$, $\gamma$. Since $\Omega_\alpha\Omega_\beta$ is a translation, if follows that if we can find a line $m$ perpendicular to $l$ such that $\Omega_\alpha\Omega_\beta = \Omega_m\Omega_\gamma$, then $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_m$ as required, since $\Omega_\gamma^{-1} = \Omega_\gamma$.

Now, if $P$ is a point on $\ell$, $N$ is a unit vector perpendicular to $\ell$, and $m$ is any line perpendicular to $\ell$, then we may choose $a, b, c$ and $d$ so that

$P + aN \in \alpha$ and $P + bN \in \beta$ so $\Omega_\alpha\Omega_\beta(x) = x + 2(a - b)N$

$P + dN \in m$ and $P + cN \in \gamma$ so $\Omega_m\Omega_\gamma(x) = x + 2(d - c)N$

To obtain $\Omega_\alpha\Omega_\beta = \Omega_m\Omega_\gamma$ we must choose $d$ such that $2(a - b) = 2(d - c)$, i.e. $d = a - b + c$.

Therefore $m = P + (a - b + c)N$ which completes the proof. \qed

**Corollary 4.12.** If $T = \Omega_\alpha\Omega_\beta$ is a translation along a line $\ell$, and $m$ is any line perpendicular to $\ell$, then there exists lines $n$ and $n'$ such that

$T = \Omega_\alpha\Omega_\beta = \Omega_m\Omega_n = \Omega_{n'}\Omega_m$.

**Proof.** Apply the previous theorem to $\Omega_m\Omega_\alpha\Omega_\beta$ to find a line $n$ such that $\Omega_m\Omega_\alpha\Omega_\beta = \Omega_n$. Then $\Omega_\alpha\Omega_\beta = \Omega_m\Omega_n$. Similarly for $\Omega_\alpha\Omega_\beta\Omega_m$. \qed
Theorem 4.13 (Three Reflection Theorem for concurrent lines). Let \( \alpha, \beta \) and \( \gamma \) be lines intersecting in a point \( P \). There exists a line \( \delta \) through \( P \) such that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta \).

Proof (Version 1). Case 1: \( P = (0,0) \). Let \( \theta, \phi, \psi \) be such that \( \Omega_\alpha = \text{ref}(\theta) \) and \( \Omega_\beta = \text{ref}(\phi) \) and \( \Omega_\gamma = \text{ref}(\psi) \). We may calculate directly (by matrix multiplication)

\[
\Omega_\alpha \Omega_\beta \Omega_\gamma = \text{ref}(\theta - \phi + \psi) = \Omega_{\theta - \phi + \psi},
\]

a reflection in the line through the origin with direction vector \((\cos(\theta - \phi + \psi), \sin(\theta - \phi + \psi))\).

Case 2: \( P \neq (0,0) \). Now, \( \tau_{-P} \Omega_\alpha \tau_P = \text{ref}(\theta) \), \( \tau_{-P} \Omega_\beta \tau_P = \text{ref}(\phi) \) and \( \tau_{-P} \Omega_\gamma \tau_P = \text{ref}(\psi) \) are all reflections in lines through the origin. Thus

\[
(\tau_{-P} \Omega_\alpha \tau_P)(\tau_{-P} \Omega_\beta \tau_P)(\tau_{-P} \Omega_\gamma \tau_P) = \tau_{-P} \Omega_\alpha \Omega_\beta \Omega_\gamma \tau_P = \text{ref}(\theta - \phi + \gamma),
\]

and so \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \tau_P(\text{ref}(\theta - \phi + \gamma)) \tau_{-P} \), a reflection in the line through \( P \) with direction \((\cos(\theta - \phi + \gamma), \sin(\theta - \phi + \gamma))\). \( \square \)

Proof (Version 2). If \( P \) is the origin, then the reflections are linear transformations and the determinant of each matrix is \(-1\) so the determinant of the product is \(-1\) and hence a reflection (because rotations have determinant \(+1\)). Of course, we haven’t said anything about determinants, so to make this into a real proof one would first have to do the work on determinants. For \( P \) not the origin the proof is as for Case 2 above. \( \square \)

Proof (Version 3). Again we start with \( P \) the origin. We know that if \( \Omega_\alpha = \text{ref}(\theta) \) and \( \Omega_\beta = \text{ref}(\phi) \) and \( \Omega_\gamma = \text{ref}(\psi) \), then \( \Omega_\alpha \Omega_\beta = \text{rot}(2(\theta - \phi)) \) is a rotation. If \( \delta \) is any line through the origin with direction \((\cos \varpi, \sin \varpi) \), then \( \Omega_\delta \Omega_\gamma = \text{rot}(2(\varpi - \psi)) \). Setting \( \delta = \theta - \phi + \varpi \) gives \( \Omega_\alpha \Omega_\beta = \Omega_\delta \Omega_\gamma \). Multiplying both sides on the right by \( \Omega_\gamma \), completes the proof. As before the case for \( P \) not the origin is as for Case 2 in Version 1. \( \square \)

Corollary 4.14. Let \( T = \Omega_\alpha \Omega_\beta \) be a rotation about point \( P \) and let \( \ell \) be any line through \( P \). There exist lines \( m \) and \( m' \) through \( P \) with \( T = \Omega_\ell \Omega_m = \Omega_m \Omega_\ell \).

The proof is essentially the same as that of Corollary 4.12.

Theorem 4.15. Let \( \Omega_\alpha, \Omega_\beta, \Omega_\gamma \) be three distinct reflections not all parallel or concurrent. Then \( \Omega_\alpha \Omega_\beta \Omega_\gamma \) is a glide reflection.

Note: A glide reflecton is a reflection followed by a translation along the line of reflection, i.e. \( \tau_v \Omega_n \) where \( v \) is parallel to \( n \).

Proof. Case 1: \( \alpha \) and \( \beta \) intersect in \( P \). We shall apply Corollary 4.14 to construct a line \( n \) and lines \( m, m' \) perpendicular to \( n \) so that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \Omega_{m'} \Omega_n \), a glide reflection as required.

Let \( \ell \) be the line through \( P \) perpendicular to \( \gamma \) and let \( F \) be the intersection of \( \ell \) and \( \gamma \). Next, using Corollary 4.14, choose a line \( m \) through \( P \) so that \( \Omega_\alpha \Omega_\beta = \Omega_m \Omega_\ell \) and hence

\[
\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \Omega_\ell \Omega_\gamma.
\]
Let \( n \) be the line through \( F \) perpendicular to \( m \) and let \( m' \) be the line through \( F \) perpendicular to \( n \). Now, \( n \) and \( m' \) are perpendicular and intersect in \( F \), \( \ell \) and \( \gamma \) are perpendicular and intersect in \( F \), so they are both half turns and hence represent the same isometry (see exercises). Thus, \( \Omega_{m'}\Omega_n = \Omega_\ell\Omega_\gamma \) and therefore

\[
\Omega_{\alpha}\Omega_{\beta}\Omega_\gamma = \Omega_{m'}\Omega_m\Omega_n ,
\]
a glide reflection.

\[
\begin{array}{ccccc}
m & & \ell & & \\
& & \alpha & & \beta \\
m' & & & P \\
& & \gamma \\
& & & n \\
& & F
\end{array}
\]

Case 2: \( \beta \) and \( \gamma \) intersect. Apply Case 1 to \( (\Omega_{\alpha}\Omega_{\beta}\Omega_\gamma)^{-1} = \Omega_\gamma\Omega_\beta\Omega_\alpha = \tau_v\Omega_n \) to obtain \( \Omega_\gamma\Omega_\beta\Omega_\alpha = \tau_v\Omega_n \) and hence

\[
\Omega_{\alpha}\Omega_{\beta}\Omega_\gamma = (\tau_v\Omega_n)^{-1} = \Omega_n\tau_{-v} = \tau_{-v}\Omega_n ,
\]
again a glide reflection. \( \square \)

**Homework Due Wednesday, October 17 2001.** Please hand this in neatly typed, with proper proofs, good spelling and decent grammar. If it isn’t typed, then it will not be accepted.

4.1 Recall that if \( \ell \) is a line, \( X \) is any point, \( P \) a point on \( \ell \), \( F \) the point at the foot of the perpendicular from \( X \) to \( \ell \) and \( N \) is a unit normal to \( \ell \), then \( F = X - \langle X - P, N \rangle N \) and \( d(X, F) = |\langle X - P, N \rangle| \).

Also note that \( F \) has the property that if \( Q \) is any other point on \( \ell \), then \( d(X, F) < d(Q, X) \). Therefore we may define \( d(X, \ell) = |\langle X - P, N \rangle| \).

Show that if \( N' \) is another unit normal to \( \ell \) and \( P' \in \ell \) then \( |\langle X - P', N' \rangle| = |\langle X - P, N \rangle| \).

What is the significance of this?

4.2 The distance between two parallel lines \( \ell, m \) is the unique number \( d(\ell, m) \) such that

\[
d(x, m) = d(y, \ell) = d(\ell, m) \quad \forall x \in \ell, y \in M .
\]
(i) Prove that if \( N \) is a unit normal to \( \ell \), then \( d(\ell, m) = |\langle x - y, N \rangle| \) for any \( x \in \ell, y \in m \).

(ii) Let \( n = \{ \frac{1}{2}(x + y) : x \in \ell, y \in m \} \). Show that \( n \) is a line parallel to \( \ell \) and \( m \) and midway between them, i.e. \( d(n, m) = d(n, \ell) \) and \( d(n, m) + d(n, \ell) = d(m, \ell) \).

4.3 Recall \( \Omega_{\ell}x = x - 2\langle x - P, N \rangle N \) where \( N \) is a unit normal to \( \ell \) and \( P \) is a point on \( \ell \). Prove that \( \langle x - P', N' \rangle N' = \langle x - P, N \rangle N \) for any other unit normal \( N' \) to \( \ell \) and point \( P' \) on \( \ell \). What does this show?

4.4 Let \( \ell = P + [v], m = Q + [v] \) be lines and \( |v| = 1 \). Show \( \Omega_{\ell}\Omega_{m} = \tau_{w} \) where \( w = 2(P - Q, v^\perp)v^\perp \) and \( \tau_{-w} = \Omega_{m}\Omega_{\ell} \). What does this show?

4.5 A Half Turn is a rotation \( \Omega_{\ell}\Omega_{m} \) where \( \ell \) is perpendicular to \( m \).

(i) Show that \( \Omega_{\alpha} \) and \( \Omega_{\beta} \) commute if and only if \( \alpha \) is perpendicular to \( \beta \).

(ii) If \( \ell, m, \alpha, \beta \) intersect at point \( P \) and \( \ell \) is perpendicular to \( m \), \( \alpha \) is perpendicular to \( \beta \), then \( \Omega_{\ell}\Omega_{m} = \Omega_{\alpha}\Omega_{\beta} \).

(iii) Show that the half turn \( H_{P} \) about point \( P \) is given by \( H_{P}X = -X + 2P \) (\( X \in \mathbb{E}^{2} \)).

(iv) Show that the product of 2 half turns is a translation along the line joining the centers of rotations.

**Theorem 4.16.** If \( T \) is a glide reflection and \( \Omega_{\alpha} \) is a reflection, then \( \Omega_{\alpha}T \) is a translation or a rotation.

**Proof.** A glide reflection is of the form \( \Omega_{\ell}\tau_{v} = \tau_{v}\Omega_{\ell} \) where \( v \) is parallel to \( \ell \).

Case 1: \( \alpha \) is parallel to \( \ell \). Now, \( \Omega_{\alpha}T = \Omega_{\alpha}\Omega_{\ell}\tau_{v} = \tau_{u}\tau_{v} = \tau_{u+v} \) since \( \tau_{u} = \Omega_{\alpha}\Omega_{\ell} \) is a translation.

Case 2: \( \alpha \) intersects \( \ell \) at point \( P \). Choose \( m, n \) perpendicular to \( \ell \) so that \( P \in m \) and \( \tau_{v} = \Omega_{m}\Omega_{n} \).

Since \( \alpha, \ell \) and \( m \) intersect in \( P \) it follows from the three reflections theorem for concurrent lines that there exists a line \( \delta \) such that \( \Omega_{\alpha}\Omega_{\ell}\Omega_{m} = \Omega_{\delta} \). Now,

\[
\Omega_{\alpha}T = \Omega_{\alpha}\Omega_{\ell}\tau_{v} = \Omega_{\alpha}\Omega_{\ell}\Omega_{m}\Omega_{n} = \Omega_{\delta}\Omega_{n} ,
\]

a rotation or translation. \( \square \)

**Theorem 4.17.** Every isometry of \( \mathbb{E}^{2} \) may be written as the product of at most three reflections.

**Proof.** Recall that all isometries may be written as one of the following:

(i) \( \text{ref}(\theta) \);

(ii) \( \text{rot}(\theta) \);

(iii) \( \tau_{v}\text{ref}(\theta)\tau_{-v} \);

(iv) \( \tau_{v}\text{rot}(\theta)\tau_{-v} \);

(v) \( \tau_{v}\text{ref}(\theta) \);

(vi) \( \tau_{v}\text{rot}(\theta) \).

Thus it is sufficient to show that the theorem is true for each of these. (i) and (ii) are special cases of (v) and (vi), and (v) may be written directly as the product of three reflections. This leaves us with (iii), (iv) and (vi).
For (iii) let $\Omega_\alpha = \text{ref}(\theta)$ and $\tau_v = \Omega_m\Omega_n$ where $0 \in n$ then

$$\tau_v \text{ref}(\theta)\tau_{-v} = \Omega_m\Omega_n\Omega_\alpha\Omega_n\Omega_m = \Omega_m\Omega_\delta\Omega_m,$$

where $\Omega_n\Omega_\alpha\Omega_n = \Omega_\delta$ for some line $\delta$ since these three lines intersect at the origin.

Next, let $\text{rot}(\theta) = \Omega_\alpha\Omega_\beta$ and $\tau_v = \Omega_m\Omega_n$ where $\alpha$ and $\beta$ intersect in the origin and $0 \in n$. Again $\Omega_n\Omega_\alpha\Omega_\beta = \Omega_\delta$ for some line $\delta$ since these three lines intersect at the origin. Thus

$$\tau_v \text{rot}(\theta)\tau_{-v} = \Omega_m\Omega_n\Omega_\alpha\Omega_\beta\Omega_n\Omega_m = \Omega_m\Omega_\delta\Omega_n\Omega_m.$$

Now, if $m$ is parallel to $\delta$, then $\Omega_m\Omega_\delta = \tau_w$ for some $w$ and $\Omega_m\Omega_\delta\Omega_n\Omega_m = \tau_w\tau_{-v} = \tau_{w-v}$, a translation which may be written as the product of two reflections.

Otherwise $m$ and $\delta$ intersect in $P$, and let $\tau_{-v} = \Omega_n\Omega_{n'}$ where $P \in n'$. By the three reflections theorem for concurrent lines there exists a line $\gamma$ through $P$ with $\Omega_m\Omega_\delta\Omega_{n'} = \Omega_\gamma$. Thus

$$\Omega_m\Omega_\delta\Omega_n\Omega_m = \Omega_m\Omega_\delta\tau_{-v} = \Omega_m\Omega_\delta\Omega_{n'}\Omega_{n'} = \Omega_\gamma\Omega_{n'},$$

the product of two reflections. This completes the proof for (iv).

For the final case (vi) let $\text{rot}(\theta) = \Omega_\alpha\Omega_\beta$ and $\tau_v = \Omega_m\Omega_n$ where $0 \in n$. There exists a line $\gamma$ through the origin with $\Omega_n\Omega_\alpha\Omega_\beta = \Omega_\gamma$ and hence

$$\tau_v \text{rot}(\theta) = \Omega_m\Omega_n\Omega_\alpha\Omega_\beta = \Omega_m\Omega_\gamma,$$

the product of two reflections as required. $\square$

### 5. Affine Transformations

**Definition 5.1.** A collineation is a bijection (i.e., both 1-1 and onto) $T : \mathbb{E}^2 \to \mathbb{E}^2$ such that if points $p, q, r$ are collinear then so are $Tp, Tq, Tr$.

**Definition 5.2.** An affine transformation is a map $T : \mathbb{E}^2 \to \mathbb{E}^2$ of the form $Tx = Ax + b$ where $A$ is a non-singular $2 \times 2$ matrix and $b$ is a vector in $\mathbb{R}^2$.

**Definition 5.3.** A map $T : V \to W$ between two vector spaces is linear if

1. $T(u + v) = Tu + Tv$, for all $u, v \in V$.
2. $T(\lambda u) = \lambda Tu$, for all scalars $\lambda$ and $u \in V$.

**Example:** the isometries of the triangle $\triangle ABC$ are all linear maps.

The set of all 2D invertible linear maps is a group which we denote by $\text{GL}(2)$.

Note that affine transformations are not, in general, linear as an affine transformation has both a linear and a translation component.

**Theorem 5.4.** All collineations are affine transformations and vice versa. In other words, Definitions 5.1 and 5.2 are equivalent.

How do isometries fit in all of this? All isometries are affine (but the converse is not true). To see this it suffices to show that reflections are affine as every isometry can be expressed as the composition of three reflections. Consider a reflection through a line $\ell$ through $p = (p_x, p_y)$ with
unit normal \( n = (n_x, n_y) \). For any point \( q = (x, y) \) \( Tq = q - 2\langle q - p, n \rangle n \) which has the form \( Tq = Aq + b \),

\[
T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1 - 2n_x^2) & (-2n_x n_y) \\ (-2n_x n_y) & (1 - 2n_y^2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \langle p, n \rangle \begin{bmatrix} n_x \\ n_y \end{bmatrix}
\]

Furthermore \( A \) is non-singular as \( \det A = (1 - 2n_x^2)(1 - 2n_y^2) - 4n_x^2 n_y^2 = 1 - 2(n_x^2 + n_y^2) = -1 \).

**Theorem 5.5.** Every isometry is an affine transformation.

**Theorem 5.6.** Let \( \text{AF}(2) \) denote the set of affine transformations on \( \mathbb{E}^2 \), then \( \text{AF}(2) \) is a group under composition.

**Proof.** Let \( Tx = Ax + c \) and \( Sx = Bx + d \) be affine transformations. Then \( T Sx = T(Bx + d) = A(Bx + d) + c = (AB)x + (Ad + c) = Mx + f \). Since \( M^{-1} = B^{-1}A^{-1} \), \( TS \) is affine as well, so \( \text{AF}(2) \) is closed under composition. The identity matrix is the identity of the group, and \( T^{-1}x = A^{-1}(x - c) = A^{-1}x - A^{-1}c \) □

**Definition 5.7.** A direction \([v]\) is the set of vectors proportional (i.e., scalar multiples) to a non-zero vector \( v \). Thus, \([v] = \{tv : t \in \mathbb{R}\}, v \in \mathbb{R}^2 \setminus \{0\}\).

**Definition 5.8.** For any point \( P \in \mathbb{R}^2 \) and direction \([v]\) the line through \( P \) with direction \([v]\) is the set \( \ell = \{x \in \mathbb{E}^2, x - P \in [v]\} \). We write \( \ell = P + [v] \).

**Theorem 5.9.** Let \( T \) be an affine transformation such that \( Tx = Ax + b \) and \( \ell = P + [v] \) a line in \( \mathbb{E}^2 \). Then \( T\ell \) is also a line in \( \mathbb{E}^2 \).

**Proof.** A point on \( \ell \) has the form \( p + tv \). Thus, \( T(p + tv) = A(p + tv) + b = (Ap + b) + tAv = Tp + t(Av) \in TP + [Av] \). In other words, \( T\ell \) is the line through \( Tp \) with direction \([Av]\). □

**Corollary 5.10.** Let \( Tx = Ax + b \) be an affine transformation. A line \( P + [v] \) is a fixed line of \( T \) if and only if the following two conditions hold,

(i) \( Av = \lambda v \) for some \( \lambda \in \mathbb{R} \setminus \{0\} \) (i.e. \( v \) is an eigenvector of \( A \)).

(ii) \( (A - I)P + b \in [v] \).

**Proof.** We saw in Theorem 5.9 that if \( \ell = P + [v] \), then \( T\ell = TP + [Av] \). Thus, \( T\ell = \ell \) if and only if \([Av] = [v]\) (i.e. \( Av = \lambda v \) for some \( \lambda \in \mathbb{R} \setminus \{0\} \)) and \( TP \in \ell \), that is \( TP = P + tv \) for some \( t \in \mathbb{R} \setminus \{0\} \). But \( TP = AP + b \) which gives \((A - I)P + b = tv \in [v]\). □

**Definition 5.11.** The set of lines perpendicular to a fixed line \( \ell \) in \( \mathbb{E}^2 \) is called a pencil of parallels.

**Theorem 5.12.** Let \( T \) be an affine transformation, then

(i) if two fixed lines of \( T \) intersect, then they do so in a fixed point of \( T \);

(ii) if two fixed lines of \( T \) are parallel, then every line in a pencil containing these lines is fixed by \( T \);

(iii) if two lines are parallel, then their images in \( T \) are parallel.
Proof. Let lines \( \ell \) and \( m \) intersect in the point \( P \) and be fixed by \( T \). Then \( TP \in T\ell = \ell \) and \( TP \in Tm = m \), so that \( TP \) is on both \( \ell \) and \( m \) and hence in their intersection. Thus \( TP = P \) which proves (i).

Let distinct lines \( \ell \) and \( m \) be parallel with direction \([v]\). Let \( Tx = Ax + b \) for \( x \in \mathbf{E}^2 \) and let \( X \in \mathbf{E}^2 \). Since \( T \) fixes \( \ell \) and \( m \), it follows from Corollary 5.10 that \( Av = \lambda v \) for some \( \lambda \in \mathbf{R} \setminus \{0\} \). Again, by Corollary 5.10 we must show that \( (A - I)X + b \in [v] \) to prove (ii). From Section 4 it is clear that we may find points \( P \in \ell \) and \( Q \in m \) such that \( Q - P \) is perpendicular to \( v \) and \( X \) lies on the line through \( P \) and \( Q \). Since \( \ell \neq m \), it follows that \( P \neq Q \) and hence we may write \( X = tP + (1 - t)Q \) for some \( t \in \mathbf{R} \). Now, by Corollary 5.10 \((A - I)P + b, (A - I)Q + b \in [v]\), thus \[
\begin{align*}
(A - I)X + b &= (A - I)(tP + (1 - t)Q) + b = t(A - I)P + tb + (1 - t)(A - I)Q + (1 - t)b \\
&= t((A - I)P + b) + (1 - t)((A - I)Q + b) \in [v],
\end{align*}
\] which completes the proof of (ii).

For (iii) we observe that if \( \ell \) and \( m \) are parallel lines with direction \([v]\), then both \( T\ell \) and \( Tm \) have direction \([Av]\). \( \square \)

We have defined an affine transformation to be a bijection \( T : \mathbf{E}^2 \to \mathbf{E}^2 \) such that \( Tx = Ax + b \) (\( x \in \mathbf{E}^2 \)) where \( A \in \mathbf{GL}(2) \) (i.e. a \( 2 \times 2 \) invertible matrix) and \( b \in \mathbf{R}^2 \). We may also represent \( T \) as a linear transformation in \( \mathbf{R}^3 \) that fixes the plane \( z = 1 \). If \( A \) and \( b \) are given as
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},
\]
then we may write \( y = Ax + b \) as
\[
\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix},
\]
and composition in \( \mathbf{AF}(2) \) is equivalent to multiplication of \( 3 \times 3 \) matrices. Thus \( \mathbf{AF}(2) \) is a subgroup of \( \mathbf{GL}(3) \), the General Linear group on \( \mathbf{R}^3 \), consisting of \( 3 \times 3 \) invertible matrices.

**Theorem 5.13.** Let \( P, Q \) be points in \( \mathbf{E}^2 \) and \( T \) an affine transformation. Then

(i) \( T((1 - t)P + tQ) = (1 - t)TP + tTQ \forall t \in \mathbf{R} \), i.e. affine transformations preserve order along lines, and

(ii) a point \( X \) lies between \( P \) and \( Q \) if and only if \( TX \) lies between \( TP \) and \( TQ \), furthermore
\[
\frac{d(P, X)}{d(Q, X)} = \frac{d(TP, TX)}{d(TQ, TX)}.
\]

**Proof.** We can calculate directly
\[
T((1 - t)P + tQ) = A((1 - t)P + tQ) + b = (1 - t)AP + (1 - t)b + tAQ + tb
\]
\[
= (1 - t)(AP + b) + t(AQ + b) = (1 - t)TP + tTQ,
\]
as required for (i).
The point $X$ lies between $P$ and $Q$ if, and only if, the exists $t \in [0,1]$ with $X = (1-t)P + tQ$. It is thus clear from (i) that $TX$ lies between $TP$ and $TQ$. Now,
\[
\frac{d(P,X)}{d(Q,X)} = \frac{|tP - tQ|}{|(1-t)P - (1-t)Q|} = \frac{t|P - Q|}{(1-t)|P - Q|} = \frac{t}{1-t}
\]
and a similar argument gives the same value for $d(TP,TX)/d(TQ,TX)$. \hfill \Box

**Theorem 5.14.** Let $T$ be an affine transformation.

(i) If $T$ fixes two distinct points, then $T$ fixes every point on the line through these two points.

(ii) If $T$ fixes three non-collinear points, then $T$ is the identity.

*Proof.* (i) is immediate from part (i) of the previous theorem. For (ii) let $T$ be an affine transformation and let $P, Q, R$ be three non-collinear points fixed by $T$.

Let $X \in \mathbb{E}^2$. If $X$ is on a line through any of the pairs of points $P, Q, R$, then $X$ is fixed by (i). Otherwise let $A$ be the midpoint of the line from $P$ to $Q$. The line through $X$ and $A$ cannot be parallel to both lines $\overline{PR}$ and $\overline{QR}$, thus it meets one of these in $B$ distinct from $A$.

\[
\begin{array}{cccc}
Q & X \\
R & A \\
B & P
\end{array}
\]

Now both $A$ and $B$ are fixed by $T$ and $X$ is on a line through $A$ and $B$. Thus $X$ is fixed using part (i). \hfill \Box

**Theorem 5.15** (Fundamental Theorem of Affine Transformations). Given 2 non-collinear triples $P, Q, R$ and $P', Q', R'$, there exists a unique affine transformation $T$ such that $TP = P'$, $TQ = Q'$ and $TR = R'$.

*Proof.* Since $\{Q - P, R - P\}$ and $\{Q' - P', R' - P'\}$ are bases for $\mathbb{R}^2$ there is a $2 \times 2$ invertible matrix $A$ such that $A(Q - P) = (Q' - P')$ and $A(R - P) = (R' - P')$ (see Lemma 5.16 below). Recall that if $B \in \mathbb{E}^2$, then $\tau_B$ is the isometry $T_BX = X + B$ and let $T = \tau_{P'}A\tau_{-P}$, then it is easy to check that $TP = P'$, $TQ = Q'$ and $TR = R'$. Hence such a transformation exists.

To show the $T$ is unique, suppose $S$ is an affine transformation that agrees with $T$ on $P, Q, R$. But now $S^{-1}T$ is affine and fixes three non-collinear points $P, Q, R$. Hence $S^{-1}T = I$ and so $S = T$ as required.

**Lemma 5.16.** Let $\{u, v\}$ and $\{x, y\}$ be two pairs of non-parallel vectors in $\mathbb{R}^2$. Then there exists an invertible $2 \times 2$ matrix $A$ such that $Au = x$ and $Av = y$.

*Proof.* Let $i = (1, 0), j = (0, 1)$, then it suffices to show that for any pair $\{u, v\}$ of non-parallel vectors there exists an invertible $2 \times 2$ matrix $A$ such that $Ai = u$ and $Aj = v$.
Let \( u = (a, b), v = (c, d) \), then a simple calculation shows that
\[
A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}
\]
is the required matrix. We only need check that \( A \) is invertible, i.e. \( \det(A) \neq 0 \). If \( \det(A) = 0 \), then \( ad - bc = 0 \). If \( a \neq 0 \), then \( d = bc/a \) and \( v = (c/a)u \), a contradiction. Otherwise, if \( a = 0 \), then since \( b \neq 0 \) it follows that \( c = 0 \) and \( v = (d/b)u \), again a contradiction. □

6. Geometry of the Sphere \( S^2 \)

6.1. Preliminaries in \( E^3 \). We need to include the properties of the cross product in \( R^3 \) since we shall use this heavily in \( S^2 \).

**Definition 6.1** (Cross Product in \( R^3 \)). The cross product of two vectors \( u, v \) in \( R^3 \) is the unique vector \( z = u \times v \in R^3 \) satisfying
\[
\langle z, x \rangle = \det(x, u, v) \text{ for every } x \in R^3.
\]

In fact we may calculate \( u \times v \) directly from \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) as
\[
u \times v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = (u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1)\]

One should check that the cross product is well defined; we leave this and the rest of the proof of the next theorem as an exercise.

**Theorem 6.2.** The cross product has the following properties:

(i) \( u \times v \) is well defined;
(ii) \( \langle u \times v, u \rangle = 0 = \langle u \times v, v \rangle \);
(iii) \( u \times v = -v \times u \);
(iv) \( \langle u \times v, w \rangle = \langle u, v \times w \rangle \);
(v) \( u \times v) \times w = u \times (v \times w) = \langle u, w \rangle v - \langle v, w \rangle u \);
(vi) \( u \times v = 0 \) if and only if \( u \parallel v \);
(vii) \( u \times v \neq 0 \) if and only if \{ \( u, v, u \times v \) \} is a basis of \( R^3 \);
(viii) \( |u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2 \).

**Definition 6.3.** A triple \( \{u, v, w\} \) of a mutually orthogonal unit vectors is called an orthonormal triple.

**Proposition 6.4.** If \( \{u, v, w\} \) is an orthonormal triple then
\[
x = \langle x, u \rangle u + \langle x, v \rangle v + \langle x, w \rangle w \text{ for every } x \in R^3.
\]

**Proposition 6.5.** If \( u \) is any unit vector then there exist \( v, w \) in \( R^3 \) such that \( \{u, v, w\} \) is an orthonormal triple.

**Definition 6.6.** A plane in \( E^3 \) is a set of points \( \prod \) satisfying:
From Theorem 6.2 (v) (previous theorem there is a unique line through \( P, Q, R \) are both perpendicular to \( Q \) and hence \( u \) and hence \( u \) are poles of \( \ell \).)

**Corollary 6.14.** Let \( v, w \in \mathbb{R}^3 \), then \([v, w] = \{su + tw : t, s \in \mathbb{R}\}\) is the span of \( v, w \).

**Theorem 6.8.** We may represent a plane in the following ways:

(i) If \( v, w \in \mathbb{R}^3 \) are non parallel vectors and \( P \in \mathbb{E}^3 \) then \( P + [v, w] \) is a plane. It is “the plane through \( P \) spanned by \( v \) and \( w \).”

(ii) If \( N \in \mathbb{R}^3 \) is a unit vector and \( P \in \mathbb{E}^3 \) then \( \{x \in \mathbb{R}^3 : \langle x - P, N \rangle = 0\} \) is a plane. It is “the plane through \( P \) with normal vector \( N \).”

(iii) If \( P, Q, R \) are non-colinear points in \( \mathbb{E}^3 \), then there is a unique plane \( \prod \) containing \( P, Q, R \). This plane is called “the plane \( P, Q, R \).”

6.2. Lines in \( S^2 \). We next want to define what a line is in \( S^2 \). If you start at any point of \( S^2 \) and move “straight ahead” then you will trace out a “great circle” and end up back where you started. In terms of the surrounding context, this is the intersection of a plane in \( \mathbb{R}^3 \) through the origin with \( S^2 \). These will be our lines.

**Definition 6.9.** A line in \( S^2 \) is the set \( \ell = \{x \in S^2 : \langle u, x \rangle = 0\} \) for some unit vector \( u \) in \( \mathbb{R}^3 \). We refer to \( u \) as the pole of \( \ell \).

Notice that the line \( \ell \) with pole \( u \) is the intersection of the plane through the origin in \( \mathbb{R}^3 \) having normal vector \( u \) with \( S^2 \).

**Definition 6.10.** Two points \( P \) and \( Q \) of \( S^2 \) are **antipodal** if \( P = -Q \).

**Theorem 6.11.** If \( u \) is a pole of \( \ell \), then so is \( -u \). If \( P \) lies on \( \ell \), then so does \( -P \).

**Theorem 6.12.** If \( P \) and \( Q \) are distinct points that are not antipodal, then there is a unique line containing \( P \) and \( Q \).

**Proof.** (i) Existence: let \( \ell \) be the line with pole \( P \times Q/|P \times Q| \). Both \( P \) and \( Q \) lie on \( \ell \) since \( P \) and \( Q \) are both perpendicular to \( P \times Q \).

(ii) Uniqueness: Let \( u \) be a pole of any line through \( P \) and \( Q \), then \( \langle P, u \rangle = 0 \) and \( \langle Q, u \rangle = 0 \). From Theorem 6.2 (v) \( P \times Q \times u = \langle P, u \rangle Q - \langle Q, u \rangle P = 0 \), i.e. \( u \) is parallel to \( P \times Q \). But since \( |u| = 1 \) this means \( u = \pm(P \times Q)/|P \times Q| \), in other words \( u \) is a pole of the line \( \ell \) from part (i). \( \square \)

**Theorem 6.13.** Let \( \ell, m \) be distinct lines in \( S^2 \), then they have exactly 2 points of intersection and these points are antipodal.

**Proof.** Let \( u, v \) be poles of \( \ell, m \) respectively. The two lines are distinct so \( u, v \) are not parallel and hence \( u \neq \pm v \), or \( u \times v \neq 0 \). Points \( \pm(u \times v)/|u \times v| \) are on both \( \ell \) and \( m \) since they are perpendicular to both \( u \) and \( v \). Thus \( \ell, m \) have at least 2 points of intersection.

If \( P \) is a third point of intersection, then \( P \) is not antipodal to either \( \pm(u \times v)/|u \times v| \) so by the previous theorem there is a unique line through \( (u \times v)/|u \times v| \) and \( P \), a contradiction. \( \square \)

**Corollary 6.14.** No two lines of \( S^2 \) are parallel.
Note: even lines with a common perpendicular intersect.

**Definition 6.15.** The distance between two points \( P, Q \in \mathbb{S}^2 \) is given by 
\[
d(P, Q) = \cos^{-1} \langle P, Q \rangle
\]

**Remark:** The range of the inverse cosine function is \([0, \pi]\) so that \(0 \leq d(P, Q) \leq \pi\).

**Theorem 6.16.** Let \( P, Q, R \in \mathbb{S}^2 \), then

(i) \( d(P, Q) \geq 0\);

(ii) \( d(P, Q) = 0 \) if and only if \( P = Q \);

(iii) \( d(P, Q) = d(Q, P) \);

(iv) \( d(P, Q) + d(Q, R) \geq d(P, R) \).

**Proof.** We shall prove (iv). Recall that \( |u \times v|^2 = |u|^2|v|^2 - \langle u, v \rangle^2 \) (Theorem 6.2). Thus, \( \langle u, v \rangle^2 \leq |u|^2|v|^2 \). Now, let \( r = d(P, Q), p = d(Q, R), q = d(P, R) \), then 
\[
\langle P \times R, Q \times R \rangle^2 \leq |P \times R|^2|Q \times R|^2.
\]
Calculating the left and right sides separately gives
\[
\langle P \times R, Q \times R \rangle^2 = \langle P, R \times Q \times R \rangle^2
\]
\[
= \langle P, ((R, R) Q - (Q, R) R) \rangle^2
\]
\[
= ((R, R) \langle P, Q \rangle - (Q, R) \langle P, R \rangle)^2
\]
\[
= (\cos r - \cos q \cos p)^2,
\]
and
\[
|P \times R|^2|Q \times R|^2 = (|P|^2|R|^2 - \langle P, R \rangle^2)(|Q|^2|R|^2 - \langle Q, R \rangle^2)
\]
\[
= (1 - \cos^2 q)(1 - \cos^2 p)
\]
\[
= \sin^2 q \sin^2 p.
\]
Thus \( (\cos r - \cos q \cos p)^2 \leq \sin^2 q \sin^2 p \) and hence \( \cos r - \cos q \cos p \leq \sin q \sin p \) since \( 0 \leq p, q \leq \pi \). This gives \( \cos r \leq \sin q \sin p + \cos q \cos p \) and then \( \cos r \leq \cos(q - p) \). Since \( \cos x \) is decreasing on \([0, \pi]\), we obtain \( r \geq q - p \) provided that \( 0 \leq q - p \leq \pi \), i.e. \( r + p \geq q \). Since \( 0 \leq p, q \leq \pi \), clearly \( p - q \leq \pi \). If \( q - p \leq 0 \leq r \) then \( q \leq p + r \) as required. \(\square\)

**Corollary 6.17.** If \( d(P, Q) + d(Q, R) = d(P, R) \), then \( P, Q \) and \( R \) are collinear.

**Proof.** An inspection of the proof of the theorem shows that if \( d(P, R) = d(P, Q) + d(Q, R) \), then 
\[
\langle P \times R, Q \times R \rangle^2 = |P \times R|^2|Q \times R|^2.
\]
From Theorem 6.2 (vii) \( \langle u, v \rangle^2 = |u|^2|v|^2 - |u \times v|^2 \) thus 
\[
\langle P \times R \rangle \times (Q \times R) = 0.
\]
If \( P \times R = 0 \) then \( P, R \) are parallel, so they must be equal or antipodal and \( Q \) is on some line between them. Similarly if \( Q \times R = 0 \), then \( P \) is on some line between \( Q \) and \( R \).

Otherwise \( P \times R \) and \( Q \times R \) are parallel. Now, the line containing \( P \) and \( R \) has a pole \( P \times R/|P \times R| \) and the line containing \( Q \) and \( R \) has a pole \( Q \times R/|Q \times R| \). Since these are parallel, the lines \( PR \), \( QR \) have the same poles and are therefore equal. \(\square\)

In fact more is true. We shall shortly prove that if equality holds, then \( Q \) lies on a half line between \( P \) and \( R \).
**Definition 6.18.** Two lines are perpendicular if their poles are perpendicular.

**Theorem 6.19.** Let \( \ell \) and \( m \) be distinct lines in \( S^2 \), then there is a unique line \( n \) such that both \( \ell \) and \( m \) are perpendicular to \( n \). Moreover, the intersection points of \( \ell \) and \( m \) are the poles of \( n \).

**Proof.** Let \( \ell \) have pole \( u \) and \( m \) have pole \( v \). As in Theorem 6.13 the point \( P = u \times v / |u \times v| \) is in the intersection of \( \ell \) and \( m \). Let \( n \) be the line with pole \( P \), then both \( u \) and \( v \) are perpendicular to \( P \), and hence \( \ell \) and \( m \) are perpendicular to \( n \). \( \square \)

**Theorem 6.20.** Let \( \ell \) be a line in \( S^2 \), \( P \in S^2 \). If \( P \) is not a pole of \( \ell \), then there exists a unique line \( m \) through \( P \) perpendicular to \( \ell \).

**Proof.** Let \( \ell \) have pole \( u \), let \( v = (u \times P) / |u \times P| \) and let \( m \) be the line with pole \( v \). Since \( P \) is perpendicular to \( v \) it follows that \( P \in m \) and \( m \) is perpendicular to \( \ell \) since \( u \) is perpendicular to \( v \).

Now let \( n \) be a different line perpendicular to \( \ell \) containing \( P \). Thus both \( m \) and \( n \) are perpendicular to \( \ell \) and hence their points of intersection are the poles of \( \ell \). Since \( P \) is on both \( m \) and \( n \), it follows that \( P \) is a pole of \( \ell \). Finally, if \( P \) is not a pole of \( \ell \), the line \( m \) must be unique. \( \square \)

### 6.3. Isometries of \( S^2 \)

We would like to define reflection in the line \( \ell \) of \( S^2 \) with pole \( u \), as \( \Omega_{\ell}x = x - 2(x,u)u \). We must first check that the range of \( \Omega_{\ell} \) is at least contained in \( S^2 \):

\[
|\Omega_{\ell}x|^2 = (x - 2(x,u)u, x - 2(x,u)u) = (x,x) - 4(x,u)^2 + 4(x,u)^2 = 1,
\]

so \( \Omega_{\ell} : S^2 \rightarrow S^2 \). Furthermore,

\[
\Omega_{\ell}\Omega_{\ell}x = \Omega_{\ell}x - 2\langle \Omega_{\ell}x, u \rangle u
\]

\[
= x - 2(x,u)u - 2(x,u)u,u)
\]

\[
= x - 2(x,u)u - 2(x,u)u + 4(x,u)u
\]

\[
= x.
\]

Thus \( \Omega_{\ell} \) is a good candidate for reflection. In fact, more is true.

**Theorem 6.21.** Let \( n \in \mathbf{R}^3 \) be a unit vector, let \( T : \mathbf{R}^3 \rightarrow \mathbf{R}^3 \) be \( Tx = x - 2(x,n)n \), then

(i) \( T \) is linear;

(ii) \( \langle Tx,Ty \rangle = \langle x,y \rangle \) for all \( x, y \in \mathbf{R}^3 \).

**Proof.** In order to show linearity we must prove \( T(x + y) = Tx + Ty \) and \( T(ax) = aTx \) for every \( x, y \in \mathbf{R}^3 \) and \( a \in \mathbf{R} \).

\[
T(x + y) = (x + y) - 2((x + y),n)n = (x - 2(x,n)n) + (y - 2(y,n)n) = Tx + Ty,
\]

and

\[
T(ax) = ax - 2(ax,n)n = a(x - 2(x,n)n) = aTx,
\]

as required.

For the second part,

\[
\langle Tx,Ty \rangle = (x - 2(x,n)n, y - 2(y,n)n) = \langle x,y \rangle - 4\langle x,n \rangle \langle y,n \rangle + 4\langle x,n \rangle \langle y,n \rangle = \langle x,y \rangle.
\]
Corollary 6.22. Let \( \Omega_\ell x = x - 2\langle x, u \rangle u \) for \( x \in S^2 \), where \( u \) is a pole of \( \ell \), then

(i) \( \Omega_\ell : S^2 \to S^2 \)
(ii) \( d(\Omega_\ell x, \Omega_\ell y) = d(x, y) \) for all \( x, y \in S^2 \)
(iii) \( \Omega_\ell \Omega_\ell x = x \) for all \( x \in S^2 \)
(iv) \( \Omega_\ell \) is a bijection.
(v) \( \Omega_\ell x = x \) if and only if \( x \in \ell \)

Proof. For \( x \in S^2 \), \( 1 = |x| = \sqrt{\langle x, x \rangle} = \sqrt{\langle \Omega_\ell x, \Omega_\ell x \rangle} = |\Omega_\ell x| \), so \( \Omega_\ell x \in S^2 \). Part (ii) is immediate from the linearity of \( \Omega_\ell \) and (iii) we have already calculated.

That \( \Omega_\ell \) is one-to-one is a consequence of part (ii), and for \( x \in S^2 \) let \( y = \Omega_\ell x \), then \( \Omega_\ell y = x \) and hence \( \Omega_\ell \) is also onto.

Finally, if \( \Omega_\ell x = x \) and \( n \) is a pole of \( \ell \), then \( x = x - 2\langle x, n \rangle n \) which forces \( \langle x, n \rangle = 0 \) since \( n \neq 0 \). In other words, \( x \) is perpendicular to a pole of \( \ell \) and so \( x \in \ell \).

Definition 6.23. A rotation about a point \( P \) is the composition of two reflections in lines \( \alpha \) and \( \beta \) that intersect in \( P \).

Theorem 6.24. If \( \alpha \), \( \beta \), and \( \gamma \) are lines that intersect in a point \( P \), then there exists a line \( \delta \) through \( P \) such that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta \).

Theorem 6.25. If \( \alpha \), \( \beta \), and \( \gamma \) are lines that intersect in a point \( P \) then there exists lines \( \delta \) and \( \delta' \) such that \( \Omega_\alpha \Omega_\beta = \Omega_\delta \Omega_\gamma = \Omega_\gamma \Omega_\delta' \).

Definition 6.26. A translation along the line \( \ell \) is \( \Omega_\alpha \Omega_\beta \) where \( \alpha \) and \( \beta \) are lines perpendicular to \( \ell \).

Note: If \( \alpha \) and \( \beta \) are distinct lines then there is a unique line \( \ell \), perpendicular to both \( \alpha \) and \( \beta \) which intersect in the poles of \( \ell \). So, a translation along \( \ell \) is the same as a rotation about a pole of \( \ell \). The same theorems hold for translations as reflections.

Theorem 6.27. Every isometry of \( S^2 \) may be written as the composition of at most three reflections.

Theorem 6.28. The composition of two rotations of \( S^2 \) is another rotation.

Theorem 6.29. Every rotation may be written as the composition of two half-turns.

6.4. Segments in \( S^2 \). In order to discuss angles and triangles in \( S^2 \), we first need some more information on lines in \( S^2 \). We may parameterize a line \( \ell \) in \( S^2 \) as follows. Let \( \ell \) have pole \( u \) and let \( P \) and \( Q \) be points chosen such that \( \{u, P, Q\} \) is orthonormal (of course, then \( P \) and \( Q \) are on \( \ell \)) and set \( \alpha(t) = (\cos t)P + (\sin t)Q \).

Theorem 6.30. Let \( \ell, u, P, Q \) be as above, then

(i) \( \ell = \{\alpha(t) : t \in \mathbb{R}\} \);
(ii) Each point of \( \ell \) occurs exactly once as \( \alpha(t) \) for \( t \in [0, 2\pi) \), i.e. \( \ell = \{\alpha(t) : t \in [0, 2\pi)\} \);
(iii) \( d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2| \) if \( 0 \leq |t_1 - t_2| \leq \pi \).
**Definition 6.31.** A subset $s$ of $S^2$ is called a **segment** if there exist points $P$ and $Q$ with $P$ perpendicular to $Q$ (i.e. $\langle P, Q \rangle = 0$) and numbers $t_1 < t_2$ with $(t_2 - t_1) < 2\pi$ such that $s = \{(\cos t)P + (\sin t)Q : t_1 \leq t \leq t_2\}$ and $\alpha(t) = (\cos t)P + (\sin t)Q$.

Note that $t_1, t_2, P, Q$ are not unique. However, if the segment $s$ may also be represented as $s = \{(\cos t)P' + (\sin t)Q' : t_1' \leq t \leq t_2'\}$ and $\alpha'(t) = (\cos t)P' + (\sin t)Q'$, then

- $|t_1 - t_2| = |t_1' - t_2'|$ (the same length),
- $\{\alpha(t_1), \alpha(t_2)\} = \{\alpha'(t_1'), \alpha'(t_2')\}$ (the same end points) and
- $P \times Q = \pm P' \times Q'$ (the line containing the segment is unique).

**Theorem 6.32.** Let $A$ and $B$ be any perpendicular points (i.e. $\langle A, B \rangle = 0$) in $S^2$. Let $s$ be a segment on line $AB$, then there exists a unique interval $[a, b]$ with $0 \leq a < 2\pi$ and $s = \{(\cos t)A + (\sin t)B : a \leq t \leq b\}$.

**Theorem 6.33.** Let $A$ and $B$ be non-antipodal points, then there exist exactly two segments with $A$ and $B$ as their endpoints. The union of the two segments is the line $AB$. The intersection is the set $\{A, B\}$.

**Definition 6.34.** The longer segment is the **major segment**, the shorter segment is the **minor segment**. If $A$ and $B$ are antipodal, then each of the (infinitely many) segments, with $A$ and $B$ as endpoints, is called a **half-line**.

The length of the minor segment $AB$ is $d(A, B)$, the length of the major segment $AB$ is $2\pi - d(A, B)$ and the length of the half-line $AB$ is $\pi$.

**Theorem 6.35.** Let $P, Q, X$ be distinct points in $S^2$. If $P$ and $Q$ are not antipodal, then $X$ lies on minor $PQ$ if and only if $d(P, X) + d(X, Q) = d(P, Q)$.

**Proof.** Let $P'$ be perpendicular to $P$ and write minor $PQ = \{(\cos t)P + (\sin t)P' : 0 \leq t \leq L\}$ where $L = d(P, Q)$. If $X$ lies on minor $PQ$, then $X = (\cos \phi)P + (\sin \phi)P'$ for some $0 < \phi < L$.

Now, $d(P, X) = \cos^{-1}(P, X) = \cos^{-1}(\cos \phi) = \phi$, and $d(X, Q) = \cos^{-1}(X, Q) = \cos^{-1}(\cos L \cos \phi + \sin L \sin \phi) = \cos^{-1}(\cos(L - \phi)) = L - \phi$. Thus $d(P, X) + d(X, Q) = \phi + L - \phi = L = d(P, Q)$ as required.

To prove the converse let $X = (\cos \phi)P + (\sin \phi)P'$ for some $-\pi < \phi < \pi$. We shall consider four cases:

- $[\phi \text{ between } P \text{ and } Q] \ 0 < \phi < L$, $d(P, X) = \phi$, $d(X, Q) = L - \phi$ and $d(P, X) + d(X, Q) = \phi + L - \phi = L = d(P, Q)$.
- $[\phi \text{ between } Q \text{ and } -P] \ L < \phi \leq \pi$, $d(P, X) = \phi$, $d(X, Q) = \phi - L$ and $d(P, X) + d(X, Q) = 2\phi - L \neq L$ since $\phi \neq L$.
- $[\phi \text{ between } P \text{ and } -Q] \ L - \pi \leq \phi < 0$, $d(P, X) = -\phi$, $d(X, Q) = L - \phi$ and $d(P, X) + d(X, Q) = L - 2\phi \neq L$ since $\phi \neq 0$.
- $[\phi \text{ between } -Q \text{ and } -P] \ -\pi < \phi < L - \pi$, $d(P, X) = -\phi$, $d(X, Q) = 2\pi + \phi - L$ and $d(P, X) + d(X, Q) = 2\pi - L \neq L$ since $L \neq \pi$. □
Definition 6.36. A ray is a half-line with one end point removed. The other end point is called the origin of the ray.

Suppose \( P \) and \( Q \) are points, and minor \( PQ \) has length \( L \) and is written minor \( PQ = \{(\cos t)P + (\sin t)P' : 0 \leq t \leq L\} \). Then \( \{(\cos t)P + (\sin t)P' : 0 \leq t < \pi\} \) is the unique ray through \( Q \) with origin \( P \). We write this ray as \( \overrightarrow{PQ} \) (which is unique).

Definition 6.37. The union of two rays \( r_1 \) and \( r_2 \) with common origin \( P \) is called an angle with vertex \( P \) and arms \( r_1 \) and \( r_2 \). If \( r_1 = r_2 \) it is the zero angle. If \( r_1 \) and \( r_2 \) are on the same line and \( r_1 \neq r_2 \) it is the straight angle.

The angle with vertex \( Q \) and arms \( QP \) and \( QR \) is written as \( \angle PQR \).

Definition 6.38. Let \( \angle PQR \) be an angle. A point \( X \) is in the interior of \( \angle PQR \) if minor \( XP \) does not intersect the line \( QR \) and minor \( XR \) does not intersect the line \( QP \). The interior is called the lune of \( \angle PQR \).

Definition 6.39. The radian measure of \( \angle PQR \) is

\[
\cos^{-1}\left(\frac{Q \times P}{|Q \times P|}, \frac{Q \times R}{|Q \times R|}\right).
\]

Definition 6.40 (Triangle). Let \( P, Q, R \) be non-collinear points the triangle \( \triangle PQR \) is the union of the minor segments \( PQ, QR, PR \) these are the sides of the triangle the interior of the triangle is the set of points \( X \) which are in the interior of each of the three angles.

There are other possible definitions for a triangle, but this has the following properties:

- \( \triangle PQR \) lies in a “hemisphere”.
- 3 non-collinear points determine a unique triangle.
- \( \pi < \text{angle sum} < 2\pi \)

Spherical trigonometry. Let \( ABC \) be a triangle sides of length \( a = d(B,C) \), \( b = d(A,C) \) and \( c = d(A,B) \), so that \( a = \cos^{-1}\langle B,C \rangle \) etc.

In the plane law of cosines

\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc}
\]

We shall use the equation \( \langle A \times B, A \times C \rangle = \langle B, C \rangle - \langle A, C \rangle \langle A, B \rangle \) from Theorem 6.2(ix) to obtain a version of this for the sphere. First note from Theorem 6.2(viii) that \( |B \times C|^2 = |B|^2 |C|^2 - \langle B, C \rangle^2 = 1 - \cos^2 a = \sin^2 a \) so that \( B \times C = \sin a \). Now,

\[
\langle B, C \rangle - \langle A, C \rangle \langle A, B \rangle = \cos a - \cos b \cos c
\]

and from the definition of angle we have

\[
\cos A = \left\langle A \times B, \frac{A \times C}{|A \times C|} \right\rangle.
\]

Thus,

\[
\cos A = \frac{\langle A \times B, A \times C \rangle}{\sin b \sin c} = \frac{\cos a - \cos b \cos c}{\sin b \sin c},
\]

the law of cosines on the sphere.
We next investigate the law of sines. Rearranging the above we obtain
\[
1 - \cos A = \frac{\cos(b - c) - \cos a}{\sin b \sin c} = 2 \sin^2(A/2)
\]
\[
1 + \cos A = \frac{\cos a - \cos(b + c)}{\sin b \sin c} = 2 \cos^2(A/2)
\]
and if we multiply these together we get
\[
\sin^2 A = \frac{(\cos(b - c) - \cos a)(\cos a - \cos(b + c))}{\sin^2 b \sin^2 c} = \frac{K}{\sin^2 b \sin^2 c}
\]
where
\[
K = 4 \sin((b - c + a)/2) \sin((b - c - a)/2) \sin((a + b + c)/2) \sin((a - b - c)/2).
\]
Setting \(a + b + c = 2s\) and dividing both sides by \(\sin^2 a\) we see that
\[
\frac{\sin^2 A}{\sin^2 a} = \frac{4 \sin(s) \sin(s - a) \sin(s - b) \sin(s - c)}{\sin^2 a \sin^2 b \sin^2 c},
\]
and since the right hand side is symmetric in \(a, b, c\) this gives
\[
\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.
\]
We must now justify our choice of \(K\): we must show that \(K = (\cos(b - c) - \cos a)(\cos a - \cos(b + c))\).
This follows from the basic trigonometric identities; we give the proof that \(\cos a - \cos(b + c) = -2 \sin((a + b + c)/2) \sin((a - b - c)/2)\) and leave the reader to show that \(\cos(b - c) - \cos a = -2 \sin((b - c + a)/2) \sin((b - c - a)/2)\).

\[
\cos a - \cos(b + c) = 2 \left( \cos^2(a/2) - \cos^2 \left( \frac{b + c}{2} \right) \right)
\]
\[
= 2 \left( \cos^2(a/2) - \cos^2 \left( \frac{b + c}{2} \right) \right)
\]
\[
= 2 \left( \cos^2(a/2) - \cos^2 \left( \frac{b + c}{2} \right) \right)
\]
\[
= 2 \left( \cos^2(a/2) \sin^2 \left( \frac{b + c}{2} \right) + \cos^2(a/2) \cos^2 \left( \frac{b + c}{2} \right) - \cos^2 \left( \frac{b + c}{2} \right) \right)
\]
\[
= 2 \left( \cos^2(a/2) \sin^2 \left( \frac{b + c}{2} \right) - \cos^2(a/2) \cos^2 \left( \frac{b + c}{2} \right) \right)
\]
\[
= -2 \left( \sin \left( \frac{b + c}{2} \right) \cos(a/2) + \sin(a/2) \cos \left( \frac{b + c}{2} \right) \right)
\]
\[
= -2 \sin \left( \frac{a + b + c}{2} \right) \sin \left( \frac{a - b - c}{2} \right)
\]

We have the following analogue of Pythagoras’s Theorem for right triangles.

**Theorem 6.41.** Let \(ABC\) be a triangle with sides of lengths \(a, b, c\) as above. If \(AC\) is perpendicular to \(AB\), then \(\cos a = \cos b \cos c\).
Proof. Let $u$ be a pole of $AB$ such that $C = (\cos b)A + (\sin b)u$. Then $B = (\cos c)A \pm (\sin c)u \times A$ and hence

$$\cos a = \cos(d(B, C)) = \cos(\cos^{-1}(B, C)) = (B, C)$$

$$= ((\cos b)A + (\sin b)u, (\cos c)A \pm (\sin c)u \times A) = \cos b \cos c.$$ 

\[\Box\]

**Theorem 6.42.** Let $L$ be a line $X$ a point not on $L$ nor a pole of $L$, let $m$ be the line through $X$ perpendicular to $L$ let $F$ be the point in $L \cap m$ closest to $X$ and $-F$ its antipode, then for every $Y$ on $L$ except $\pm F$ we have $d(X, F) < d(X, Y) < d(X, -F)$.

Proof. Let $X = C$, $F = A$, $Y = B$ in the previous theorem, so that $a = d(X, Y)$, $b = d(X, F)$, $c = d(F, Y)$, $d(X, -F) = \pi - b$ and $b < \pi/2$. Thus we have $\cos a = \cos b \cos c$ and since $b < \pi/2$, $\cos b > 0$ so that $\cos a$ and $\cos c$ have same sign. If both are positive, then $\cos a = \cos b \cos c < \cos b$ since $\cos c < 1$ and $\cos b > 0$. Now, cosine is decreasing on $[0, \pi]$ which gives $b < a < \pi/2 < \pi - b$. If both are negative, then $b < \pi/2 < a$ and $\cos(\pi - a) = \cos b \cos(\pi - c) < \cos b$ (since $\cos(\pi - \theta) = -\cos \theta$) and as before $b < \pi - a$ so that $b < a < \pi - b$. If both zero, then $b < \pi/2 = a < \pi - b$ \[\Box\]

**Definition 6.43.** In the notation of the previous theorem $F$ is the foot of the perpendicular from $X$ to $L$, and $d(X, F)$ is written as $d(X, L)$, the distance from $X$ to $L$.

**Concurrence theorems.** We have the same results as in $E^2$ for the intersection of the perpendicular bisectors of the sides of a triangle, and the intersection of the angle bisectors of the angles of a triangle.

**Theorem 6.44.** The perpendicular bisectors of the three sides of a triangle are concurrent.

The proof is the same as in $E^2$.

Proof. Let $\Delta PQR$ be the triangle and let $M$ be the intersection of the perpendicular bisectors of $PQ$ and $QR$. Now, $d(P, M) = d(Q, M)$ and $d(Q, M) = d(R, M)$ so that $d(P, M) = d(R, M)$ and hence $M$ is also on the perpendicular bisector of $PR$. \[\Box\]

**Theorem 6.45.** Let $P$, $Q$ and $R$ be non-collinear in $S^2$, let $p = QR$, $q = PR$, $r = PQ$ and let reflections $\Omega_u$ interchange $p$, $q$, and $\Omega_v$ interchange $q$, $r$. Then there exists a line $w$ concurrent with $u$, $v$ such that $\Omega_w$ interchanges $p$, $r$.

Proof. Let $M$ be the intersection of $u$, $v$ and let $\ell$ be the line through $M$ perpendicular to $q$. By the three reflection theorem there exists a line $w$ through $M$ with $\Omega_u \Omega_q \Omega_u = \Omega_w$. Now, $\Omega_w p = \Omega_u \Omega_q \Omega_u p = \Omega_u \Omega q = \Omega u q = r$, as required. \[\Box\]

**Corollary 6.46.** The lines bisecting the three angles of a triangle are concurrent.

Proof. Exercise. Note that there is something to prove, since (see below) there are at least two reflections interchanging any pair of lines. \[\Box\]
Lemma 6.47. Given any pair of distinct, non-parallel, lines \(p, q\) in \(S^2\), there exist exactly two reflections that interchange them.

Proof. Exercise. \(\square\)

7. The Projective Plane \(P^2\)

Definition 7.1. The Projective plane \(P^2\) can be defined in the following three ways.

1. \(P^2\) is the set of all pairs \(\{x, -x\}\) of antipodal points in \(S^2\).
2. \(P^2\) is the set of lines through the origin \(O\) in \(E^3\).
3. \(P^2\) is the set of equivalence classes of triples \((x_1, x_2, x_3)\) where \(x_i \in \mathbb{R}\), not all zero (i.e. non-zero vectors in \(R^3\), where vectors are equivalent if they are parallel).

We can represent the same point in \(P^2\) as any useful multiple of \((x_1, x_2, x_3)\). For example, \((1, 1, 1)\) and \((2, 2, 2)\) represent the same point.

Let \(\pi : S^2 \rightarrow P^2\) be the mapping \(\pi : x \mapsto \{x, -x\}\) (projection). \(\pi\) is a two-to-one map from \(S^2\) to \(P^2\).

Definition 7.2. A line of \(P^2\) is a set \(\pi \ell\) where \(\ell\) is a line in \(S^2\). If \(u\) is a pole of \(\ell\), then \(\pi u\) is the pole of \(\pi \ell\).

Theorem 7.3.

(i) Two distinct lines in \(P^2\) intersect in exactly one point.

(ii) Two distinct points in \(P^2\) determine a unique line.

Proof. Let \(\ell\) and \(m\) be distinct lines in \(S^2\) and let \(P \in \ell, Q \in m\) be such that \(\pi P = \pi Q\). Then \(P = \pm Q\) and since \(\pi Q = \pi (-Q)\) we may assume \(P = Q\). Thus a point of intersection of \(\pi \ell\) and \(\pi m\) must be the projection of a point of intersection of \(\ell\) and \(m\). Let \(u, v\) be the poles of \(\ell\) and \(m\) respectively. Since \(\ell\) and \(m\) are distinct it follows that \(u, v\) are not antipodal and \(\pm u \times v/|u \times v|\) are the two points of intersection of \(\ell\) and \(m\). But \(\pi(u \times v/|u \times v|) = \pi(-u \times v/|u \times v|)\) are the same point in \(P^2\).

Let \(\pi x, \pi y\) be distinct in \(P^2\), then \(x, y\) are not antipodal in \(S^2\). Thus there exists a unique line \(\ell\) containing \(x, y\) and so \(\pi x, \pi y \in \pi \ell\). \(\square\)

7.1. Homogeneous Coordinates.

Definition 7.4. Let \(\{e_1, e_2, e_3\}\) be a basis for \(R^3\). Then each \(x \in R^3\) determines a unique triple \((x_1, x_2, x_3)\) by \(x = x_1 e_1 + x_2 e_2 + x_3 e_3\). If \(\pi x \in P^2\), \(\lambda \in R \setminus \{0\}\) and \(\lambda x = t_1 e_1 + t_2 e_2 + t_3 e_3\), then \((t_1, t_2, t_3)\) is called a homogeneous coordinate vector of \(\pi x\) and \(t_1, t_2, t_3\) are called the homogeneous coordinates of \(\pi x\).

Theorem 7.5. Let \(P, Q, R, S\) be distinct points in \(P^2\), such that no three are collinear. Then there exists a basis of \(R^3\) with respect to which the four points have homogeneous coordinate vectors \((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\).
Proof. Let \( v_1, v_2, v_3 \) in \( \mathbb{R}^3 \) be representatives of \( P, Q, R \). Since \( P, Q, R \) are not collinear, \( \{v_1, v_2, v_3\} \) are linearly independent and hence a basis for \( \mathbb{R}^3 \). If \( v_4 \) is any representative of \( S \), then \( v_4 = k_1v_1 + k_2v_2 + k_3v_3 \), for some \( k_1, k_2, k_3 \in \mathbb{R} \). Set \( e_1 = k_1v_1, e_2 = k_2v_2, e_3 = k_3v_3 \), then \( \{e_1, e_2, e_3\} \) is the required basis.

\[ \text{Theorem 7.6. Let } x \text{ and } y \text{ be homogeneous coordinate vectors of two distinct points in } \mathbb{P}^2. \text{ Then } \lambda x + \mu y, \text{ for } \lambda, \mu \in \mathbb{R} \text{ is a typical point on the line they determine.} \]

Proof. \( \mathbb{P}^2 \) is the set of equivalence classes of the set \( \{(x, y, z) : x, y, z \in \mathbb{R} \text{ not all zero}\} \) under the equivalence relation \( \sim \), where \( (x, y, z) \sim (u, v, w) \) if there exists \( t \in \mathbb{R} \setminus \{0\} \) with \( u = tx, v = ty, w = tz \). A homogeneous coordinate vector is any equivalence class representative.

Let the basis for the homogeneous coordinate vectors be \( \{e_1, e_2, e_3\} \) and let \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \). Given vectors \( u = x_1e_1 + x_2e_2 + x_3e_3, v = y_1e_1 + y_2e_2 + y_3e_3 \) in \( \mathbb{R}^3 \), the plane in \( \mathbb{R}^3 \) containing \( u \) and \( v \) is given by \( \{\lambda u + \mu v : \lambda, \mu \in \mathbb{R}\} \). Lines in \( \mathbb{P}^2 \) correspond to planes in \( \mathbb{R}^3 \) through the origin \( O \). So every homogeneous coordinate vector \( \lambda x + \mu y \) corresponds to a point \( \lambda u + \mu v \) in the plane containing \( u, v \) (except \( O \)) and vice versa. \( \square \)

Note: For the moment we shall use \( \text{triangle} \) in \( \mathbb{P}^2 \) to mean three non-collinear points.

\[ \text{Theorem 7.7 (Desargues's Theorem). Let } PQR, P'Q'R' \text{ be triangles in } \mathbb{P}^2 \text{ such that the lines } PP', QQ', RR' \text{ are concurrent. Then } PQ \cap P'Q', QR \cap Q'R', PR \cap P'R' \text{ are collinear.} \]

Proof. Let \( X \) be the point of concurrence. If \( X \) is collinear with any two of \( P, Q, R \), then two lines (e.g. \( PQ, P'Q' \)) would coincide and the theorem is trivial.

Otherwise no three of \( P, Q, R, X \) are collinear and we may represent them as homogeneous coordinate vectors \( P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1), X = (1, 1, 1) \). Now, \( P' \) is on \( \overline{XP} \) so \( P' = \lambda(1, 0, 0) + \mu(1, 1, 1) = (\lambda + \mu, \mu, \mu) \), so we can write \( P' = (p, 1, 1) \), for some \( p \in \mathbb{R} \). Similarly \( Q' = (1, q, 1) \) and \( R' = (1, 1, r) \).

The equation of the line \( \overline{PQ} \) is \( x_3 = 0 \). A point on \( \overline{PQ} \) looks like \( \lambda(p, 1, 1) + \mu(1, q, 1) = (\lambda p + \mu, \lambda + \mu q, \lambda + \mu) \). So the equation is \( (1-q)x_1 + (1-p)x_2 + (pq-1)x_3 = 0 \). They intersect in the point \( L \). Now \( x_3 = 0 \) and hence \( (1-q)x_1 + (1-p)x_2 = 0 \). If \( x_1 = p-1 \), then \( x_2 = 1-q \). So \( L = (p-1, 1-q, 0) \).

Similarly \( \overline{QR} \cap \overline{Q'R'} = M \), where \( M = (0, q-1, 1-r) \) and \( \overline{PR} \cap \overline{P'R'} = N = (1-p, 0, r-1) \). Since the sum of these homogeneous coordinate vectors is 0, the points \( L, M, N \) are collinear. \( \square \)

A. APPENDIX: GROUPS

We collect here a few useful definitions of groups for completeness.

\[ \text{Definition A.1. A group is a set } G \text{ together with an operator } \circ : G \times G \to G, \text{ denoted by } (G, \circ), \text{ that satisfies:} \]

(i) Associativity: \((a \circ b) \circ c = a \circ (b \circ c)\), for all \( a, b, c \in G \)

(ii) Identity: there exists \( e \in G \) such that \( a \circ e = e \circ a = a \), for all \( a \in G \).

(iii) Inverses: for all \( a \in G \), there exists \( a^{-1} \in G \) such that \( a \circ a^{-1} = a^{-1} \circ a = e \).

\[ \text{Definition A.2. A subset } H \subset G \text{ of a group } (G, \circ) \text{ is a subgroup of } G \text{ if} \]

(i) \( a \circ b \in H, \forall a, b \in H \)
(ii) \( a^{-1} \in H, \forall a \in H \)

Equivalently, \( H \) is a subgroup of \( G \) if and only if \( a \circ b^{-1} \in H, \forall a, b \in H \).

B. Appendix: Equivalence relations

A relation \( R \) on a set \( S \) is a subset of \( S \times S = \{(x, y) : x, y \in S\} \). For example the relation \(< \) on \( \mathbb{R} \) is given by the set \( \{(x, y) : x, y \in \mathbb{R}, x < y\} \). We say \( xRy \) to mean \( X \) is related to \( y \) by \( R \).

Definition B.1. A relation \( R \) is

(i) reflexive if \( aRa \) for all \( a \in S \).
(ii) symmetric if \( aRb \) implies \( bRa \) for all \( a, b \in S \).
(iii) transitive if \( aRb \) and \( bRc \) implies \( aRc \) for all \( a, b, c \in S \).

An equivalence relation is one which satisfies all three.

Definition B.2. An equivalence class of an equivalence relation \( \sim \) is the set \([a] = \{b \in S : b \sim a\}\). The element \( a \) of \( S \) is an equivalence class representative.

For example \( a \sim b \) in \( \mathbb{Z} \) if \( a = b \mod 7 \). The equivalence classes are \([0],[1],...,[6]\).

Theorem B.3. Equivalent classes are disjoint and independent of the representative.

C. Appendix: Linear Independence of Vectors

Definition C.1. Vectors \( v_1, v_2, ..., v_n \) in \( \mathbb{R}^n \) are linearly independent if \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) implies \( a_i = 0 \) for \( i = 1, \ldots, n \).

Definition C.2. If \( v_1, v_2, ..., v_n \) are linearly independent in \( \mathbb{R}^n \), then every vector \( v \in \mathbb{R}^n \) can be written as \( v = a_1v_1 + \cdots + a_nv_n \). The set \( \{v_1, v_2, ..., v_n\} \) is a basis for \( \mathbb{R}^n \).

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