1. Reflections, Translations and Rotations in $\mathbb{E}^2$

In this section we discuss reflections: isometries of $\mathbb{E}^2$ given by reflecting in a line. The product of two reflections is either a translation or a rotation, and the product of three reflections is another reflection. In fact we shall see that any isometry may be written as the product of at most three reflections.

1.1. Reflections. In order to write down a formula for a reflection we first need to know how to reflect a point in a line.

**Theorem 1.** If $\ell$ and $m$ are perpendicular lines, then they have a unique point in common.

**Proof.** Let $\ell = P + [v]$ and $m = Q + [w]$ with $|v| = |w| = 1$. Since $\ell$ and $m$ are perpendicular it follows that $\langle v, w \rangle = 0$ and we may write

$$P - Q = (P - Q, v)v + (P - Q, w)w .$$

This gives $P - (P - Q, v)v = Q + (P - Q, w)w$ and setting $F = P - (P - Q, v)v = Q + (P - Q, w)w$ we see that $F$ is on both lines.

If there are two points $F, G$ on both $\ell$ and $m$, then $F - G \in [v] \cap [w] = \{0\}$ and hence $F - G = 0$, i.e. $F = G$. Thus the point $F$ is unique. □

The following corollary is an immediate consequence of this theorem.

**Corollary 2.** Let $X$ be a point and $\ell$ in $\mathbb{E}^2$, then there exists a unique line $m$ through $X$ perpendicular to $\ell$ and

(i) $m = X + [n]$ where $n$ is a unit normal to $\ell$ (that is, if $[v]$ is the direction of $\ell$ with $v = (v_1, v_2)$, then $n = \pm (v_2/\sqrt{v_1^2 + v_2^2}, -v_1/\sqrt{v_1^2 + v_2^2})$);

(ii) $\ell$ and $m$ intersect in $F = X - \langle X - P, n \rangle n$ where $P$ is any point on $\ell$;

(iii) $d(X, F) = |\langle X - P, n \rangle|$.

We can now use the formula for the intersection of two perpendicular lines to find the equation for the reflection of a point in the line, and thus define reflections.

**Definition 3.** The reflection of a point $X$ in a line $\ell = P + [v]$ is the point $X'$ that satisfies

$$\frac{1}{2}(X + X') = F$$

where $F = X - \langle X - P, n \rangle n$ is the intersection of $\ell$ with the line through $X$ perpendicular to $\ell$.

Now,

$$\frac{1}{2}(X + X') = F , \text{ so}$$

$$\frac{1}{2}X + \frac{1}{2}X' = X - \langle X - P, n \rangle n \text{ and hence}$$

$$\frac{1}{2}X' = \frac{1}{2}X - \langle X - P, n \rangle n \text{ thus}$$

$$X' = X - 2\langle X - P, n \rangle n .$$
Definition 4 (Reflection). The reflection in line $\ell$ (where $\ell = P + [v]$) is the bijection $\Omega_\ell : \mathbb{E}^2 \to \mathbb{E}^2$ given by

$$\Omega_\ell X = X - 2(X - P, n)n,$$

where $n$ is a unit normal vector to $\ell$.

1.2. Translations. We next investigate what happens when we compose two reflections. The result depends on whether or not the lines of reflection are parallel. In the following diagrams $X'$ is the reflection of $X$ in $\ell$, while $X''$ is the reflection of $X'$ in $m$.

- $\ell$ parallel to $m$: translation perpendicular to $\ell$
- $\ell$ not parallel to $m$: rotation about $P$

Figure 2. Reflection of $X$ in line $\ell$, then in line $m$
If \( m \) and \( n \) are parallel lines, \( P \) is on \( m \) and \( Q \) is at the foot of the perpendicular from \( P \) to \( n \), and \( N \) is the unit normal to \( m \) (and hence also to \( n \)), then

\[
\Omega_m \Omega_n X = \Omega_n X - 2(\Omega_n X - P, N)N
\]
\[
= X - 2(X - Q, N)N - 2((X - 2(X - Q, N)) - P, N)N
\]
\[
\]
\[
= X - 2(X - P, N)N + 2(X - Q, N)N
\]
\[
= X + 2(P - Q)
\]

**Figure 3.** Reflection of \( X \) in line \( n \), then in line \( m \)

**Definition 5** (Translation). Let \( \ell \) be a line and \( m, n \) lines perpendicular to \( \ell \). The transformation \( \Omega_m \Omega_n \) is a translation along \( \ell \). If \( m \neq n \), then the translation is non-trivial.

**Theorem 6.** If \( T \) is a non-trivial translation along a line \( \ell \), then \( \ell \) has a direction vector \( v \) such that \( Tx = x + v \) and conversely.

**Proof.** Let \( N \) be a unit direction vector for \( \ell \), let \( P \in \mathbf{E}^2 \) and let \( \alpha, \beta \) be distinct lines perpendicular to \( \ell \). Let \( a, b \) be the unique real numbers such that \( P + aN \in \alpha \) and \( P + bN \in \beta \). Then

\[
\Omega_a \Omega_b X = x + 2((P + aN) - (P + bN)) = x + 2(a - b)N.
\]

If \( T \) is not the identity, then \( a \neq b \) and \( 2(a - b)N \) is the required direction.

Conversely, for any \( \lambda \in \mathbf{R} \) let \( T_\lambda x = x + \lambda N \) and let \( a, b \) satisfy \( \lambda = 2(a - b) \). Setting \( \alpha = P + aN + [N^\perp] \) and \( \beta = P + bN + [N^\perp] \), where \( N^\perp \) is a unit normal vector to \( \ell \), gives \( T_\lambda = \Omega_\alpha \Omega_b \) as required. \( \square \)

**Definition 7.** For \( v \in \mathbf{R} \) let \( \tau_v \) be the translation \( \tau_v x = x + v \).
1.3. Rotations. Let $\ell = P + [v]$ be a line with $|v| = 1$. There exists a unique $\theta \in (-\pi, \pi]$ such that $v = (\cos \theta, \sin \theta)$. We shall use the unit normal to $v$, $N = (-\sin \theta, \cos \theta)$. We have shown that reflection in the line $\ell$ is given by

$$\Omega_\ell X = X - 2\langle X - P, N \rangle N.$$ 

Now let $\ell_0 = 0 + [v]$ be the line through the origin parallel to $\ell$; we shall show that $\Omega_\ell = \tau_P \Omega_{\ell_0} \tau_{-P}$. We have

$$\Omega_\ell X - P = X - P - 2\langle X - P, N \rangle N \quad \text{and} \quad \Omega_{\ell_0} X = X - 2\langle X, N \rangle N \quad \text{hence} \quad \Omega_\ell X - P = \Omega_{\ell_0} (X - P).$$

Thus $\Omega_\ell X = \Omega_{\ell_0} (X - P) + P$ as required.

Let $x = (x_1, x_2) \in \mathbb{R}^2$, then $\langle x, N \rangle = -x_1 \sin \theta + x_2 \cos \theta$ and writing the vectors in columns we have

$$\Omega_{\ell_0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2(-x_1 \sin \theta + x_2 \cos \theta) \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} (1 - 2 \sin^2 \theta)x_1 + (2 \sin \theta \cos \theta)x_2 \\ (2 \sin \theta \cos \theta)x_1 + (1 - 2 \cos^2 \theta)x_2 \end{bmatrix} = \cos 2\theta \begin{bmatrix} \sin 2\theta \\ \cos 2\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$ 

The matrix

$$\text{ref} \theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

is therefore a reflection in the line through the origin with direction $(\cos \theta, \sin \theta)$.

We saw in Figure 2 that reflection in two non-parallel lines gives a rotation. If $m$ is a second line through $P$ with direction $(\cos \phi, \sin \phi)$, then

$$\text{ref} \theta \text{ref} \phi = \begin{bmatrix} \cos 2(\theta - \phi) & -\sin 2(\theta - \phi) \\ \sin 2(\theta - \phi) & \cos 2(\theta - \phi) \end{bmatrix}.$$ 

We write a matrix of this form as

$$\text{rot} \theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$ 

Since $\text{rot} \theta$ takes the vector $(1, 0)$ and $(0, 1)$ to $(\cos \theta, \sin \theta)$ and $(-\sin \theta, \cos \theta)$ respectively, we can think of $\text{rot} \theta$ as a rotation.

**Definition 8** (Rotation). If $\alpha$ and $\beta$ are lines through a point $P$, then the isometry $\Omega_\alpha \Omega_\beta$ is called a rotation about $P$. 