**Theorem 1** (Three Reflection Theorem for parallel lines). If \( \alpha, \beta, \gamma \) are all parallel, then there exists a line \( \delta \) parallel to \( \alpha, \beta, \gamma \) such that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta. \)

**Proof.** Let \( \ell \) be a line perpendicular to \( \alpha, \beta, \gamma \). Since \( \Omega_\alpha \Omega_\beta \) is a translation, if follows that if we can find a line \( m \) perpendicular to \( \ell \) such that \( \Omega_\alpha \Omega_\beta = \Omega_m \Omega_\gamma \), then \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \) as required, since \( \Omega_m^{-1} = \Omega_\gamma \).

Now, if \( P \) is a point on \( \ell \), \( N \) is a unit vector perpendicular to \( \ell \), and \( m \) is any line perpendicular to \( \ell \), then we may choose \( a, b, c \) and \( d \) so that

\[
P + aN \in \alpha \text{ and } P + bN \in \beta \text{ so } \Omega_\alpha \Omega_\beta(x) = x + 2(a - b)N
\]

\[
P + dN \in m \text{ and } P + cN \in \gamma \text{ so } \Omega_m \Omega_\gamma(x) = x + 2(d - c)N
\]

To obtain \( \Omega_\alpha \Omega_\beta = \Omega_m \Omega_\gamma \) we must choose \( d \) such that \( 2(a - b) = 2(d - c) \), i.e. \( d = a - b + c \). Therefore \( m = P + (a - b + c)N \) which completes the proof. \( \square \)

**Corollary 2.** If \( T = \Omega_\alpha \Omega_\beta \) is a translation along a line \( \ell \), and \( m \) is any line perpendicular to \( \ell \), then there exists lines \( n \) and \( n' \) such that

\[
T = \Omega_\alpha \Omega_\beta = \Omega_m \Omega_n = \Omega_{n'} \Omega_m.
\]

**Proof.** Apply the previous theorem to \( \Omega_m \Omega_\alpha \Omega_\beta \) to find a line \( n \) such that \( \Omega_m \Omega_\alpha \Omega_\beta = \Omega_n \). Then \( \Omega_\alpha \Omega_\beta = \Omega_m \Omega_n \). Similarly for \( \Omega_\alpha \Omega_\beta \Omega_m \).

**Theorem 3** (Three Reflection Theorem for concurrent lines). Let \( \alpha, \beta \) and \( \gamma \) be lines intersecting in a point \( P \). There exists a line \( \delta \) through \( P \) such that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta \).

**Proof (Version 1).** Case 1: \( P = (0, 0) \). Let \( \theta, \phi, \psi \) be such that \( \Omega_\alpha = \text{ref}(\theta) \) and \( \Omega_\beta = \text{ref}(\phi) \) and \( \Omega_\gamma = \text{ref}(\psi) \). We may calculate directly (by matrix multiplication)

\[
\Omega_\alpha \Omega_\beta \Omega_\gamma = \text{ref}(\theta - \phi + \psi) = \Omega_{\theta - \phi + \psi},
\]

a reflection in the line through the origin with direction vector \((\cos(\theta - \phi + \psi), \sin(\theta - \phi + \psi))\).

Case 2: \( P \neq (0, 0) \). Now, \( \tau_\alpha \Omega_\alpha = \text{ref}(\theta) \), \( \tau_\beta \Omega_\beta = \text{ref}(\phi) \) and \( \tau_\gamma \Omega_\gamma = \text{ref}(\psi) \) are all reflections in lines through \( P \) so that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \tau_\alpha \Omega_\alpha \Omega_\beta \Omega_\gamma \).

By the previous theorem,

\[
(\tau_\alpha \Omega_\alpha \tau_\beta)(\tau_\alpha \Omega_\alpha \tau_\beta)(\tau_\alpha \Omega_\alpha \tau_\beta) = \tau_\alpha \Omega_\alpha \Omega_\beta \Omega_\gamma
\]

and so \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \tau_\alpha \Omega_\alpha \Omega_\beta \Omega_\gamma \), a reflection in the line through \( P \) with direction \((\cos(\theta - \phi + \gamma), \sin(\theta - \phi + \gamma))\)). \( \square \)

**Proof (Version 2).** If \( P \) is the origin, then the reflections are linear transformations and the determinant of each matrix is \(-1\) so the determinant of the product is \(-1\) and hence a reflection (because rotations have determinant \(+1\)). Of course, we haven’t said anything about determinants, so to make this into a real proof one would first have to do the work on determinants. For \( P \neq \) not the origin the proof is as for Case 2 above. \( \square \)

**Proof (Version 3).** Again we start with \( P \) the origin. We know that if \( \Omega_\alpha = \text{ref}(\theta) \) and \( \Omega_\beta = \text{ref}(\phi) \) and \( \Omega_\gamma = \text{ref}(\psi) \), then \( \Omega_\alpha \Omega_\beta = \text{rot}(2(\theta - \phi)) \) is a rotation. If \( \delta \) is any line through the origin with direction \((\cos \kappa, \sin \kappa)\), then \( \Omega_\delta \Omega_\gamma = \text{rot}(2(\kappa - \psi)) \). Setting \( \delta = \theta - \phi + \kappa \) gives \( \Omega_\alpha \Omega_\beta = \Omega_\delta \Omega_\gamma \).
Multiplying both sides on the right by $\Omega_\gamma$ completes the proof. As before the case for $P$ not the origin is as for Case 2 in Version 1.

\begin{corollary}
Let $T = \Omega_\alpha \Omega_\beta$ be a rotation about point $P$ and let $\ell$ be any line through $P$. There exist lines $m$ and $m'$ through $P$ with $T = \Omega_\ell \Omega_m = \Omega_{m'} \Omega_\ell$.

The proof is essentially the same as that of Corollary 2.
\end{corollary}

\begin{theorem}
Let $\Omega_\alpha$, $\Omega_\beta$, $\Omega_\gamma$ be three distinct reflections not all parallel or concurrent. Then $\Omega_\alpha \Omega_\beta \Omega_\gamma$ is a glide reflection.

Note: A glide reflection is a reflection followed by a translation along the line of reflection, i.e. $\tau_v \Omega_n$ where $v$ is parallel to $n$.

\begin{proof}
Case 1: $\alpha$ and $\beta$ intersect in $P$. We shall apply Corollary 4 to construct a line $n$ and lines $m, m'$ perpendicular to $n$ so that $\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \Omega_{m'} \Omega_n$, a glide reflection as required.

Let $\ell$ be the line through $P$ perpendicular to $\gamma$ and let $F$ be the intersection of $\ell$ and $\gamma$. Next, using Corollary 4, choose a line $m$ through $P$ so that $\Omega_\alpha \Omega_\beta = \Omega_m \Omega_\ell$ and hence

$$\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \Omega_\ell \Omega_n .$$

Let $n$ be the line through $F$ perpendicular to $m$ and let $m'$ be the line through $F$ perpendicular to $n$. Now, $n$ and $m'$ are perpendicular and intersect in $F$, $\ell$ and $\gamma$ are perpendicular and intersect in $F$, so they are both half turns and hence represent the same isometry (see exercises). Thus, $\Omega_m \Omega_n = \Omega_\ell \Omega_\gamma$ and therefore

$$\Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_m \Omega_{m'} \Omega_n ,$$

a glide reflection.
\end{proof}

Case 2: $\beta$ and $\gamma$ intersect. Apply Case 1 to $(\Omega_\alpha \Omega_\beta \Omega_\gamma)^{-1} = \Omega_\gamma \Omega_\beta \Omega_\alpha = \tau_v \Omega_n$ to obtain $\Omega_\gamma \Omega_\beta \Omega_\alpha = \tau_v \Omega_n$ and hence

$$\Omega_\alpha \Omega_\beta \Omega_\gamma = (\tau_v \Omega_n)^{-1} = \Omega_\alpha \tau_{-v} = \tau_{-v} \Omega_n,$$

again a glide reflection. $\square$