Homework Due Wednesday, October 17 2001. Please hand this in neatly typed, with proper proofs, good spelling and decent grammar. If it isn’t typed, then it will not be accepted.

0.1 Recall that if \( \ell \) is a line, \( X \) is any point, \( P \) a point on \( \ell \), \( F \) the point at the foot of the perpendicular from \( X \) to \( \ell \) and \( N \) is a unit normal to \( \ell \), then \( F = X - \langle X - P, N \rangle N \) and \( d(X, F) = |\langle X - P, N \rangle| \).

Also note that \( F \) has the property that if \( Q \) is any other point on \( \ell \), then \( d(X, F) < d(Q, X) \). Therefore we may define \( d(X, \ell) = |\langle X - P, N \rangle| \).

Show that if \( N' \) is another unit normal to \( \ell \) and \( P' \in \ell \) then \( |\langle X - P', N' \rangle| = |\langle X - P, N \rangle| \).

What is the significance of this?

0.2 The distance between two parallel lines \( \ell, m \) is the unique number \( d(\ell, m) \) such that

\[
d(x, m) = d(y, \ell) = d(\ell, m) \quad \forall x \in \ell, y \in M.
\]

(i) Prove that if \( N \) is a unit normal to \( \ell \), then \( d(\ell, m) = |\langle x - y, N \rangle| \) for any \( x \in \ell, y \in m \).

(ii) Let \( n = \{ \frac{1}{2}(x + y) : x \in \ell, y \in m \} \). Show that \( n \) is a line parallel to \( \ell \) and \( m \) and midway between them. i.e. \( d(n, m) = d(n, \ell) \) and \( d(n, m) + d(n, \ell) = d(m, \ell) \).

0.3 Recall \( \Omega_{x} = x - 2\langle x - P, N \rangle N \) where \( N \) is a unit normal to \( \ell \) and \( P \) is a point on \( \ell \). Prove \( \langle x - P', N' \rangle N' = \langle x - P, N \rangle N \) for any other unit normal \( N' \) to \( \ell \) and point \( P' \) on \( \ell \).

What does this show?

0.4 Let \( \ell = P + [v], \ m = Q + [v] \) be lines and \( |v| = 1 \). Show \( \Omega_{\ell} \Omega_{m} = \tau_{w} \) where \( w = 2\langle P - Q, v^{\perp}\rangle v^{\perp} \) and \( \tau_{w} = \Omega_{m} \Omega_{\ell} \).

0.5 A Half Turn is a rotation \( \Omega_{\ell} \Omega_{m} \) where \( \ell \) is perpendicular to \( m \).

(i) Show that \( \Omega_{\alpha} \) and \( \Omega_{\beta} \) commute if and only if \( \alpha \) is perpendicular to \( \beta \).

(ii) If \( \ell, m, \alpha, \beta \) intersect at point \( P \) and \( \ell \) is perpendicular to \( m \), \( \alpha \) is perpendicular to \( \beta \) then \( \Omega_{\ell} \Omega_{m} = \Omega_{\alpha} \Omega_{\beta} \).

(iii) Show that the half turn \( H_{P} \) about point \( P \) is given by \( H_{P}X = -X + 2P \) \((X \in \mathbb{E}^{2})\).

(iv) Show that the product of 2 half turns is a translation along the line joining the centers of rotations.

**Theorem 1.** If \( T \) is a glide reflection and \( \Omega_{\alpha} \) is a reflection, then \( \Omega_{\alpha} T \) is a translation or a rotation.

**Proof.** A glide reflection is of the form \( \Omega_{\ell} \tau_{v} = \tau_{v} \Omega_{\ell} \) where \( v \) is parallel to \( \ell \).

Case 1: \( \alpha \) is parallel to \( \ell \). Now, \( \Omega_{\alpha} T = \Omega_{\alpha} \Omega_{\ell} \tau_{v} = \tau_{u} \tau_{v} = \tau_{u+v} \) since \( \tau_{u} = \Omega_{\alpha} \Omega_{\ell} \) is a translation.

Case 2: \( \alpha \) intersects \( \ell \) at point \( P \). Choose \( m, n \) perpendicular to \( \ell \) so that \( P \in m \) and \( \tau_{v} = \Omega_{m} \Omega_{n} \).

Since \( \alpha, \ell \) and \( m \) intersect in \( P \) it follows from the three reflections theorem for concurrent lines that there exists a line \( \delta \) such that \( \Omega_{\alpha} \Omega_{\ell} \Omega_{m} = \Omega_{\delta} \). Now,

\[
\Omega_{\alpha} T = \Omega_{\alpha} \Omega_{\ell} \tau_{v} = \Omega_{\alpha} \Omega_{\ell} \Omega_{m} \Omega_{n} = \Omega_{\delta} \Omega_{n},
\]

a rotation or translation. \( \Box \)

**Theorem 2.** Every isometry of \( \mathbb{E}^{2} \) may be written as the product of at most three reflections.

**Proof.** Recall that all isometries may be written as one of the following:
(i) ref(θ); 
(ii) rot(θ); 
(iii) \( \tau_v \text{ref}(\theta)\tau_{-v} \); 
(iv) \( \tau_v \text{rot}(\theta)\tau_{-v} \); 
(v) \( \tau_v \text{ref}(\theta) \); 
(vi) \( \tau_v \text{rot}(\theta) \).

Thus it is sufficient to show that the theorem is true for each of these. (i) and (ii) are special 
cases of (v) and (vi), and (v) may be written directly as the product of three reflections. This 
leaves us with (iii), (iv) and (vi).

For (iii) let \( \Omega_\alpha = \text{ref}(\theta) \) and \( \tau_v = \Omega_m\Omega_n \) where \( 0 \in n \) then 
\[ \tau_v \text{ref}(\theta)\tau_{-v} = \Omega_m\Omega_n\Omega_\alpha\Omega_n\Omega_m = \Omega_m\Omega_\delta\Omega_m \ , \]
where \( \Omega_\alpha\Omega_n\Omega_m = \Omega_\delta \) for some line \( \delta \) since these three lines intersect at the origin.

Next, let rot(θ) = \( \Omega_\alpha\Omega_\beta \) and \( \tau_v = \Omega_m\Omega_n \) where \( \alpha \) and \( \beta \) intersect in the origin and \( 0 \in n \). Again 
\( \Omega_m\Omega_\alpha\Omega_\beta = \Omega_\delta \) for some line \( \delta \) since these three lines intersect at the origin. Thus 
\[ \tau_v \text{rot}(\theta)\tau_{-v} = \Omega_m\Omega_n\Omega_\alpha\Omega_\beta\Omega_n\Omega_m = \Omega_m\Omega_\delta\Omega_n\Omega_m \ . \]

Now, if \( m \) is parallel to \( \delta \), then \( \Omega_m\Omega_\delta = \tau_w \) for some \( w \) and \( \Omega_m\Omega_\delta\Omega_n\Omega_m = \tau_w\tau_{-v} = \tau_w\tau_{v} \), a 
translation which may be written as the product of two reflections.

Otherwise \( m \) and \( \delta \) intersect in \( P \), and let \( \tau_{-v} = \Omega_n\Omega_m' \) where \( P \in n' \). By the three reflections 
theorem for concurrent lines there exists a line \( \gamma \) through \( P \) with \( \Omega_n\Omega_\delta\Omega_m' = \Omega_\gamma \). Thus 
\[ \Omega_m\Omega_\delta\Omega_n\Omega_m = \Omega_m\Omega_\delta\tau_{-v} = \Omega_m\Omega_\delta\Omega_n\Omega_m = \Omega_m\Omega_\delta\Omega_n\Omega_m' = \Omega_m\Omega_\gamma \ , \]
the product of two reflections. This completes the proof for (iv).

For the final case (vi) let rot(θ) = \( \Omega_\alpha\Omega_\beta \) and \( \tau_v = \Omega_m\Omega_n \) where \( 0 \in n \). There exists a line \( \gamma \) 
through the origin with \( \Omega_n\Omega_\alpha\Omega_\beta = \Omega_\gamma \) and hence 
\[ \tau_v \text{rot}(\theta) = \Omega_m\Omega_n\Omega_\alpha\Omega_\beta = \Omega_m\Omega_\gamma \ , \]
the product of two reflections as required. □

1. Geometry of the Sphere \( S^2 \)

**Preliminaries in \( \mathbb{E}^3 \).**

**Definition 3** (Cross Product in \( \mathbb{R}^3 \)). The cross product of two vectors \( u, v \) in \( \mathbb{R} \) is the unique 
vector \( z = u \times v \in \mathbb{R}^3 \) satisfying 
\[ \langle z, x \rangle = \det(x, u, v) \quad \text{for every} \quad x \in \mathbb{R}^3 \ . \]

In fact we may calculate \( u \times v \) directly from \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) as 
\[
\begin{vmatrix}
\bar{i} & \bar{j} & \bar{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = (u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1)
\]
One should check that the cross product is well defined; we leave this and the rest of the proof of the next theorem as an exercise.

**Theorem 4.** The cross product has the following properties:

(i) \( u \times v \) is well defined;
(ii) \( \langle u \times v, u \rangle = 0 = \langle u \times v, v \rangle \);
(iii) \( u \times v = -v \times u \);
(iv) \( \langle u \times v, w \rangle = \langle u, v \times w \rangle \);
(v) \( u \times v = 0 \) if and only if \( u \parallel v \);
(vi) \( u \times v \neq 0 \) if and only if \( \{u, v, u \times v\} \) is a basis of \( \mathbb{R}^3 \);
(vii) \( |u \times v|^2 = |u|^2|v|^2 + \langle u, v \rangle^2 \).

**Definition 5.** A triple \( \{u, v, w\} \) of mutually orthogonal unit vectors is called an orthonormal triple.

**Proposition 6.** If \( \{u, v, w\} \) is an orthonormal triple then
\[
 x = \langle x, u \rangle u + \langle x, v \rangle v + \langle x, w \rangle w \quad \text{for every} \quad x \in \mathbb{R}^3
\]

**Proposition 7.** If \( u \) is any unit vector then there exist \( v, w \) in \( \mathbb{R}^3 \) such that \( \{u, v, w\} \) is an orthonormal triple.

**Definition 8.** A plane in \( \mathbb{E}^3 \) is a set of points \( \Pi \) satisfying:

(i) There is a point of \( \mathbb{E}^3 \) not in \( \Pi \).
(ii) \( \Pi \) cannot be contained in a line.
(iii) Given any two points in \( \Pi \) the line through those points is also in \( \Pi \).

**Definition 9.** Let \( v, w \in \mathbb{R}^3 \), then \( [v, w] = \{sv + tw : s, t \in \mathbb{R}\} \) is the span of \( v, w \).

**Theorem 10.** We may represent a plane in the following ways:

(i) If \( v, w \in \mathbb{R}^3 \) are non parallel vectors and \( P \in \mathbb{E}^3 \) then \( P + [v, w] \) is a plane. It is “the plane through \( P \) spanned by \( v \) and \( w \).”
(ii) If \( N \in \mathbb{R}^3 \) is a unit vector and \( P \in \mathbb{E}^3 \) then \( \{x \in \mathbb{R}^3 : \langle x - P, N \rangle = 0\} \) is a plane. It is “the plane through \( P \) with normal vector \( N \).”
(iii) If \( P, Q, R \) are non-colinear points in \( \mathbb{E}^3 \), then there is a unique plane \( \Pi \) containing \( P, Q, R \). This plane is called “the plane \( P, Q, R \).”