We next want to define what a line is in $S^2$. If you start at any point of $S^2$ and move “straight ahead” then you will trace out a “great circle” and end up back where you started. In terms of the surrounding context, this is the intersection of a plane in $\mathbb{R}^3$ through the origin with $S^2$. These will be our lines.

**Definition 1.** A line in $S^2$ is the set $\ell = \{ x \in S^2 : \langle u, x \rangle = 0 \}$ for some unit vector $u$ in $\mathbb{R}^3$. We refer to $u$ as the pole of $\ell$.

Notice that the line $\ell$ with pole $u$ is the intersection of the plane in $\mathbb{R}^3$ having normal vector $u$ with $S^2$.

**Definition 2.** Two points $P$ and $Q$ of $S^2$ are **antipodal** if $P = -Q$.

**Theorem 3.** If $u$ is a pole of $\ell$, then so is $-u$. If $P$ lies on $\ell$, then so does $-P$.

**Theorem 4.** If $P$ and $Q$ are distinct points that are not antipodal, then there is a unique line containing $P$ and $Q$.

**Proof.** (i) Existence: let $\ell$ be the line with pole $P \times Q/|P \times Q|$. Both $P$ and $Q$ lie on $\ell$ since $P$ and $Q$ are both perpendicular to $P \times Q$.

(ii) Uniqueness: Let $u$ be a pole of any line through $P$ and $Q$, then $\langle P, u \rangle = 0$ and $\langle Q, u \rangle = 0$. From above: $u \times v \times w = \langle u, w \rangle v - \langle v, w \rangle u$. So $(P \times Q) \times u = 0$, i.e. $u$ is parallel to $P \times Q$. But since $|u| = 1$ this means $u = \pm(P \times Q)/|P \times Q|$, in other words $u$ is a pole of the line $\ell$ from part (i). □

**Theorem 5.** Let $\ell, m$ be distinct lines in $S^2$, then they have exactly 2 points of intersection and these points are antipodal.

**Proof.** Let $u, v$ be poles of $\ell, m$ respectively. The two lines are distinct so $u, v$ are not parallel and hence $u \neq \pm v$, or $u \times v \neq 0$. Points $\pm(u \times v)/|u \times v|$ are on both $\ell$ and $m$ since they are perpendicular to both $u$ and $v$. Thus $\ell, m$ have at least 2 points of intersection.

If $P$ is a third point of intersection, then $P$ is not antipodal to either $\pm(u \times v)/|u \times v|$ so by the previous theorem there is a unique line through $(u \times v)/|u \times v|$ and $P$, a contradiction. □

**Corollary 6.** No two lines of $S^2$ are parallel.

Note: even lines with a common perpendicular intersect.

**Definition 7.** The **distance between two points** $P, Q \in S^2$ is given by $d(P, Q) = \cos^{-1} \langle P, Q \rangle$.

**Remark:** The range of the inverse cosine function is $[0, \pi]$ so that $0 \leq d(P, Q) \leq \pi$.

**Theorem 8.** Let $P, Q, R \in S^2$, then

(i) $d(P, Q) \geq 0$;
(ii) $d(P, Q) = 0$ if and only if $P = Q$;
(iii) $d(P, Q) = d(Q, P)$;
(iv) $d(P, Q) + d(Q, R) \geq d(P, R)$. 1
Proof. We shall prove (iv). Recall that $|u \times v|^2 = |u|^2|v|^2 - \langle u, v \rangle^2$ (Theorem 6.2). Thus, $\langle u, v \rangle^2 \leq |u|^2|v|^2$. Now, let $r = d(P, Q)$, $p = d(Q, R)$, $q = d(P, R)$, then $\langle P \times R, Q \times R \rangle^2 \leq |P \times R|^2|Q \times R|^2$. Calculating the left and right sides separately gives

\[
\langle P \times R, Q \times R \rangle^2 = \langle P, R \times Q \times R \rangle^2
\]
\[
= \langle P, (\langle R, R \rangle Q - \langle Q, R \rangle R) \rangle^2
\]
\[
= (\langle R, R \rangle \langle P, Q \rangle - \langle Q, R \rangle \langle P, R \rangle)^2
\]
\[
= (\cos r - \cos q \cos p)^2 ,
\]

and

\[
|P \times R|^2|Q \times R|^2 = (|P|^2|R|^2 - \langle P, R \rangle^2)(|Q|^2|R|^2 - \langle Q, R \rangle^2)
\]
\[
= (1 - \cos^2 q)(1 - \cos^2 p)
\]
\[
= \sin^2 q \sin^2 p .
\]

Thus $(\cos r - \cos q \cos p)^2 \leq \sin^2 q \sin^2 p$ and hence $\cos r - \cos q \cos p \leq \sin q \sin p$ since $0 \leq p, q \leq \pi$. This gives $\cos r \leq \sin q \sin p + \cos q \cos p$ and then $\cos r \leq \cos(q - p)$. Since $\cos x$ is decreasing on $[0, \pi]$, we obtain $r \geq q - p$ provided that $0 \leq q - p \leq \pi$, i.e. $r + p \geq q$. Since $0 \leq p, q \leq \pi$, clearly $p - q \leq \pi$. If $q - p \leq 0 \leq r$ then $q \leq p + r$ as required. □