CONCERNING THE BOURGAIN $\ell_1$ INDEX OF A BANACH SPACE

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Abstract. A well known argument of James yields that if a Banach space $X$ contains $\ell^n_1$'s uniformly, then $X$ contains $\ell^n_1$'s almost isometrically. In the first half of the paper we extend this idea to the ordinal $\ell_1$-indices of Bourgain. In the second half we use our results to calculate the $\ell_1$-index of certain Banach spaces. Furthermore we show that the $\ell_1$-index of a separable Banach space not containing $\ell_1$ must be of the form $\omega^\alpha$ for some countable ordinal $\alpha$.

1. Introduction

It is well known that if $\ell_p$ ($1 \leq p < \infty$) or $c_0$ is crudely finitely representable in a Banach space $X$, then it is finitely representable in $X$. This was shown for $\ell_1$ and $c_0$ by R.C. James [J] and for $\ell_p$ ($1 < p < \infty$) it is a consequence of Krivine’s theorem [K] as noted by Rosenthal [R], [L]. We may state this as

For all $p \in [1, \infty]$, every $K \geq 1$, each $m \geq 1$, and every $\varepsilon > 0$, there exists $n$ such that

if $(x_i)_1^n$ is a normalized basic sequence in a Banach space $X$ with $(x_i)_1^n K \sim uvb \ell_p^n$,

then there exists a normalized block basis $(y_i)_1^n$ of $(x_i)_1^n$ satisfying $(y_i)_1^n 1+\varepsilon \sim uvb \ell_p^n$.

Separable Banach spaces not containing $\ell_1$ may differ in the complexity of $\ell^n_1$’s embedded inside. This complexity is measured in part by Bourgain’s $\ell_1$-index [B]. Bourgain considered trees $T(X, K)$ whose nodes are finite basic sequences in the unit ball of a Banach space $X$, $K$-equivalent to the unit vector basis of some finite dimensional $\ell_1$, for a fixed $K$. The $\ell_1$-$K$-ordinal index of $X$, $I(X, K)$, was then defined to be the supremum of the orders of such trees.

1991 Mathematics Subject Classification. Primary: 46B.

Research supported by the NSF and TARP.
The definition of the $\ell_1$-trees constructed by Bourgain may be extended to $\ell_p$-trees ($1 < p \leq \infty$) (we explain all the unfamiliar terms in the next section). We extend the results on finite representability of $\ell_p$ in $X$ to $\ell_p$-trees for $p = 1$ or $\infty$. We prove the following theorem in Section 4.

**Theorem 1.1.** For $p = 1$ or $\infty$, for each $K > 1$, for every $\alpha < \omega_1$, and any $\varepsilon > 0$, there exists $\beta < \omega_1$ such that for all Banach spaces $X$, if $T$ is an $\ell_p$-tree on $X$ with constant $K$ and order, $o(T) \geq \beta$, then there exists an $\ell_p$ block subtree $T'$ of $T$ with constant $1 + \varepsilon$ and order, $o(T') \geq \alpha$.

This theorem is not true in general for $1 < p < \infty$, and in the final section we explain why not. We also show how the same ideas may be applied to the $\ell_1$-$S_\alpha$-spreading models introduced by Kiriakouli and Negrepontis [KN].

In Section 5 we apply our results to the problem of calculating Bourgain’s $\ell_1$-index $I(X)$ of certain spaces $X$. We show for example that if $X$ is Tsirelson’s space, then $I(X) = \omega^\omega$. We prove that $I(X)$ is always of the form $\omega^\alpha$ and relate $I(X)$ to the “block” Bourgain $\ell_1$-index $I_b(X)$ for spaces with a basis. Both indices are defined in Section 5.

2. Preliminaries on trees

By a tree we shall mean a countable, non-empty, partially ordered set $(T, \leq)$ for which the set \{\(y \in T : y < x\)\} is linearly ordered and finite for each $x \in T$. The elements of $T$ are called nodes. The predecessor node of $x$ is the maximal element $x'$ of the set \{\(y \in T : y < x\)\}, so that if $y < x$, then $y \leq x'$. The initial nodes of $T$ are the minimal elements of $T$ and the terminal nodes are the maximal elements. A subtree of a tree $T$ is a subset of $T$ with the induced ordering from $T$. This is clearly again a tree. Further, if $T' \subset T$ is a subtree of $T$ and $x \in T$, then we write $x < T'$ to mean $x < y$ for every $y \in T'$. We will also consider trees related to some fixed set $X$. A tree on a set $X$ is a subset $T \subseteq \cup_{n=1}^\infty X^n$ with the ordering given by: $(x_1, \ldots, x_m) \leq (y_1, \ldots, y_n)$ if $m \leq n$ and $x_i = y_i$ for $i = 1, \ldots, m$. 
The property of trees which is most interesting here is their order. Before we can define this we must recall some terminology. Let the derived tree of a tree $T$ be $D(T) = \{ x \in T : x < y \text{ for some } y \in T \}$. It is easy to see that this is simply $T$ with all of its terminal nodes removed.

We then associate a new tree $T^\alpha$ to each ordinal $\alpha$ inductively as follows. Let $T^0 = T$, then given $T^\alpha$ let $T^{\alpha+1} = D(T^\alpha)$. If $\alpha$ is a limit ordinal, and we have defined $T^\beta$ for all $\beta < \alpha$, let $T^\alpha = \cap_{\beta < \alpha} T^\beta$.

A tree $T$ is well-founded provided there exists no subset $S \subseteq T$ with $S$ linearly ordered and infinite. The order of a well-founded tree $T$ is defined as $o(T) = \inf \{ \alpha : T^\alpha = \emptyset \}$.

A tree $T$ on a topological space $X$ is said to be closed provided the set $T \cap X^n$ is closed in $X^n$, endowed with the product topology, for each $n \geq 1$. We have the following result (see [B], [D]) concerning the order of a closed tree on a Polish space.

**Proposition 2.1.** If $T$ is a well-founded, closed tree on a Polish (separable, complete, metrizable) space, then $o(T) < \omega_1$.

A map $f : T \to T'$ between trees $T$ and $T'$ is a tree isomorphism if $f$ is one to one, onto and order preserving. We will write $T \simeq T'$ if $T$ is tree isomorphic to $T'$ and $f : T \sim T'$ to denote an isomorphism. From now on we shall simply write isomorphism rather than tree isomorphism.

**Definition 2.2.** Minimal tree

A tree $t$ is a minimal tree of order $\alpha$, for some ordinal $\alpha < \omega_1$, if for each tree $T$ of order $\alpha$ there exists a subtree $T' \subset T$ of order $\alpha$ which is isomorphic to $t$. Notice that if $t$ is a minimal tree of order $\alpha$ then any subtree of $t$ of order $\alpha$ is also a minimal tree of order $\alpha$. We construct certain minimal trees for each ordinal $\alpha < \omega_1$ in Section 3.

If $X$ is a Banach space and $(x_i)_1^m \subset X$ with $\|x_i\| = 1 \ (i = 1, \ldots, m)$ we write $(x_i)_1^m \simuvb \ell_p^m$ if there exist constants $c, C$ with $c^{-1}C \leq K$ and

$$c \left( \sum_{i=1}^m |a_i|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^m a_i x_i \right\| \leq C \left( \sum_{i=1}^m |a_i|^p \right)^{\frac{1}{p}}$$
for all \((a_i)_1^m \subset \mathbb{R}\).

**Definition 2.3.** \(\ell_p\)-K-tree

An \(\ell_p\)-K-tree on a Banach space \(X\) is a tree \(T\) on \(X\) such that \(T \subseteq \bigcup_{n=1}^{\infty} S(X)^n\) and \((x_i)_1^m \sim K\) ub \(\ell_p^m\) for each \((x_1, \ldots, x_m) \in T\). We say \(T\) is an \(\ell_p\)-tree on \(X\) if \(T\) is an \(\ell_p\)-K-tree for some \(K\). For \(p = 1\) this definition is slightly different to that in \([B]\) where an \(\ell_1\)-K-tree is the largest tree of this form. In fact our trees are subtrees of those.

**Definition 2.4.** Block subtree

Let \(T\) be an \(\ell_1\)-tree on a Banach space \(X\). We say \(S\) is a block subtree of \(T\), written \(S \preceq T\), if \(S\) is a tree on \(X\) such that there exists a subtree \(T' \subset T\) and an isomorphism \(f : T' \sim S\) satisfying:

- For each \(x = (x_1, \ldots, x_n) \in T'\), let \(y = (x_1, \ldots, x_m)\) be the initial node of \(T'\) with \(y \leq x\). If \(y\) is also an initial node of \(T\), then let \(k = 0\), otherwise let \((x_1, \ldots, x_k)\) be the predecessor node of \(y\) in \(T\). Then \(f(x)\) is a normalized block basis of \((x_{k+1}, \ldots, x_n)\).

- If \(x = (x_1, \ldots, x_n) \in T'\) has predecessor node \((x_1, \ldots, x_m)\) in \(T'\), then \(f(x) = f(x') \cup (y_i)_1^k\), where \((y_i)_1^k\) is a normalized block basis of \((x_{m+1}, \ldots, x_n)\).

For each node \((y_1, \ldots, y_k) = f(x) \in S\) we call \(x\) the parent node of \((y_1, \ldots, y_k)\). Note that if \(T\) is an \(\ell_1\)-K-tree on \(X\) and \(S\) is a block subtree of \(T\) then \(S\) is also an \(\ell_1\)-K-tree on \(X\). However, if \(T\) is an \(\ell_p\)-K-tree on \(X\) for \(p > 1\) and \(S\) is a block subtree of \(T\), then \(S\) is an \(\ell_p\)-K\(^2\)-tree on \(X\).

### 3. Ordinal Trees

Most of the work needed to prove Theorem 1.1 is concerned with constructing certain general trees consisting of collections of finite subsets of ordinals ordered by inclusion. We first construct specific minimal trees \(T_\alpha\) of order \(\alpha\) for every ordinal \(\alpha < \omega_1\). Once this is done we construct “replacement trees” \(T(\alpha, \beta)\) which are formed by replacing each node of \(T_\alpha\) by one or more copies of \(T_\beta\), and show that \(T(\alpha, \beta)\) is a minimal tree of order \(\beta \cdot \alpha\). This gives us in some sense a “tree within a tree” or “an \(\alpha\) tree of \(\beta\) trees”.
These two results are used as follows: Given an arbitrary $\ell_1$-$K$-tree on $X$ with $o(T) \geq \alpha^2$ we can find a subtree isomorphic to $T(\alpha, \alpha)$. For one of the $\alpha$ trees inside this we either have a good constant—in which case we are finished—or we take a vector in the linear span of one of its nodes with a bad constant. Putting some of these vectors together yields a block subtree of order $\alpha$, each of whose nodes is "bad", and then following the original argument of James these vectors together have a good constant.

We now define the trees $T_\alpha$, and prove in Lemma 3.3 that $T_\alpha$ is minimal of order.

**Definition 3.1.** Minimal trees, $T_\alpha$

We define trees $T_\alpha$ of order $\alpha$ for each countable ordinal $\alpha$ as subsets of $[1, \gamma]^{<\omega}$ ordered by inclusion, for some ordinal $\gamma = \gamma(\alpha) < \omega_1$ where, if $S$ is any set, then $[S]^{<\omega}$ is the collection of all finite subsets of $S$. We choose $\gamma(\alpha)$ and $T_\alpha$ by induction as follows: Let $T_1 = \{\{1\}\}$. Given $T_\alpha \subset [1, \gamma]^{<\omega}$ for some ordinal $\gamma < \omega_1$, let $T_{\alpha+1} = \{A \cup \{\gamma+1\} : A \in T_\alpha\} \cup \{\{\gamma+1\}\}$. Note that for $\beta < \alpha$, $(T_{\alpha+1})^{(\beta)} = \{A \cup \{\gamma+1\} : A \in (T_\alpha)^{(\beta)}\} \cup \{\{\gamma+1\}\}$ and $(T_{\alpha+1})^{(\alpha)} = \{\{\gamma+1\}\}$. Thus $o(T_{\alpha+1}) = \alpha + 1$ as required.

Finally, to define $T_\alpha$ for $\alpha$ a limit ordinal, let $\alpha_n \nearrow \alpha$ be a sequence of ordinals increasing to $\alpha$, and let $T_{\alpha_n} \subset [1, \beta_n]^{<\omega}$ for some $\beta_n < \omega_1$. Let $\beta = \sup_n \beta_n$ and $\gamma_n = \beta + n$ for each $n$. Let $\tilde{T}_{\alpha_n} = \{A \cup \{\gamma_n\} : A \in T_{\alpha_n}\}$ and let $T_\alpha = \bigcup^\infty_1 \tilde{T}_{\alpha_n}$, ordered by inclusion. Notice that $\tilde{T}_{\alpha_n}$ is the same tree as $T_{\alpha_n}$ with the same order and structure, but the nodes have simply been relabeled. The reason for doing this is that nodes from different trees are now incomparable, and so the union $\bigcup^\infty_1 \tilde{T}_{\alpha_n}$ is a disjoint union.
To give an idea of what these trees look like we will construct the trees $T_n$ and $T_\omega$ explicitly.

$$T_1 = \{\{1\}\}$$

$$T_2 = \{\{1, 2\}, \{2\}\}$$

$$T_3 = \{\{1, 2, 3\}, \{2, 3\}, \{3\}\}$$

$$\vdots$$

$$T_n = \{\{1, 2, 3, \ldots, n\}, \{2, 3, \ldots, n\}, \ldots, \{n-1, n\}, \{n\}\}.$$  

Then to construct $T_\omega$ we use the trees $\tilde{T}_n$ ($n \geq 1$) as described above.

$$\tilde{T}_1 = \{\{1, \omega + 1\}\}$$

$$\tilde{T}_2 = \{\{1, 2, \omega + 2\}, \{2, \omega + 2\}\}$$

$$\vdots$$

$$\tilde{T}_n = \{\{1, \ldots, n, \omega + n\}, \ldots, \{n, \omega + n\}\}$$

$$T_\omega = \{\{1, \omega + 1\}, \{1, 2, \omega + 2\}, \{2, \omega + 2\}, \ldots, \{1, \ldots, n, \omega + n\}, \ldots, \{n, \omega + n\}, \ldots\}.$$  

**Lemma 3.2.** Let $\alpha < \omega_1$ be a limit ordinal and $T$ be a countable tree of order $\alpha$. Then there exist a sequence $(\alpha_n)$ of successor ordinals and a sequence $(t_n)$ of subtrees $t_n \subset T$ with $\alpha = \sup_n \alpha_n$, $o(t_n) = \alpha_n$ and $T = \cup_1^\infty t_n$. Moreover the trees $(t_n)$ are mutually incomparable, ie. if $x \in t_n$ and $y \in t_m$ with $n \neq m$, then $x$ and $y$ are incomparable.

**Proof.** Suppose that $T$ has only finitely many initial nodes; let these be $x_1, \ldots, x_n$, and let $t_i = \{y \in T : y \geq x_i\}$. Then $\alpha = o(T) = \max_{1 \leq i \leq n} o(t_i) = o(t_{i_0})$ for some $i_0 \leq n$. Let $t = \{y \in t_{i_0} : y > x_{i_0}\}$ and let $\beta = o(t)$. Since $\{x_{i_0}\}$ is the unique initial node of $t_{i_0}$, it follows that $t_{i_0} = t \cup \{x_{i_0}\}$ and hence $(t_{i_0})^\beta = \{x_{i_0}\}$. Thus $\alpha = o(t_{i_0}) = \beta + 1$, a successor ordinal, contradicting the assumption that $\alpha$ is a limit ordinal.
Thus $T$ must have infinitely many initial nodes; let these be $(x_n)^\infty_1$ and let $t_n = \{ y \in T : y \geq x_n \}$, $\alpha_n = o(t_n)$. Note that these trees are mutually incomparable since the nodes $(x_n)^\infty_1$ are incomparable. We find that $\alpha_n$ is a successor ordinal using the same argument as above and from the definition of the order of a tree we have that $o(T) = \sup_n o(t_n)$ and hence $\alpha = \sup_n \alpha_n$. □

**Lemma 3.3.** $T_\alpha$ is a minimal tree of order $\alpha$.

*Proof.* The order of $T_\alpha$ is clear from the construction; we prove here that if $T$ is any tree of order $\alpha < \omega_1$, then there exists a subtree $T' \subset T$ such that $T'$ is isomorphic to $T_\alpha$. We use induction on $\alpha$, the order of $T$. The result is obvious for $\alpha = 1$.

Suppose the lemma is true for the ordinal $\alpha < \omega_1$. Let $T$ have order $\alpha + 1$, and hence $T^\alpha \neq \emptyset$. Let $x$ be a terminal node of $T^\alpha$ and let $\tilde{T} = \{ y \in T : y > x \}$; then $o(\tilde{T}) = \alpha$. By assumption there exists a subtree $\tilde{T}'$ of $\tilde{T}$ and an isomorphism $f : T_\alpha \iso \tilde{T}'$. Clearly $T' = \tilde{T}' \cup \{ x \}$ is a subtree of $T$ of order $\alpha + 1$ and we can extend $f$ to $F : T_{\alpha + 1} \iso T'$ to show that $T'$ is isomorphic to $T_{\alpha + 1}$ as follows. Recall from Definition 3.1 that we obtain $T_{\alpha + 1}$ from $T_\alpha$ by setting $T_{\alpha + 1} = \{ a \cup \{ \gamma + 1 \} : a \in T_\alpha \cup \{ \{ \gamma + 1 \} \} \}$. Setting $F(\{ \gamma + 1 \}) = x$ and $F(a \cup \{ \gamma + 1 \}) = f(a)$ makes $F$ the required isomorphism.

If $\alpha$ is a limit ordinal, let the lemma be true for all $\beta < \alpha$ and let $T$ have order $\alpha$. By Lemma 3.2, $T = \cup_1^\infty t_n$ where $o(t_n) = \beta_n$, $\alpha = \sup_n \beta_n$, each $\beta_n$ is a successor ordinal, and the trees $(t_n)$ are mutually incomparable. Let $\alpha_n \not\nearrow \alpha$ be the sequence of ordinals increasing to $\alpha$, and let $\tilde{T}_{\alpha_n}$ be the trees, from the definition of the minimal tree $T_\alpha$, Definition 3.1. Let $(\beta_n)$ be a subsequence of $(\beta_n)$ so that $\alpha_n \leq \beta_n$ for all $n$. Each tree $t_r$ contains a subtree of order $\alpha_n$; hence, by assumption, for each $n$ there exists $t'_{r_n} \subset t_r$ and an isomorphism $f_n : T_{\alpha_n} \iso t'_{r_n}$. Using the notation of Definition 3.1 we define $\tilde{f}_n : \tilde{T}_{\alpha_n} \iso t'_{r_n}$ by $\tilde{f}_n(a \cup \{ \gamma_n \}) = f_n(a)$. Let $T' = \cup_1^\infty t'_{r_n}$ and $f : T_\alpha \iso T'$ be the function $f = \cup_1^\infty \tilde{f}_n$. □

**Remark 3.4.** It follows that if $T$ is a tree of order $\beta \geq \alpha$, then there exists a subtree $T' \subset T$ such that $T'$ is isomorphic to $T_\alpha$. 


We now construct the replacement trees $T(\alpha, \beta)$, for each pair of ordinals $\alpha, \beta < \omega_1$, promised earlier. First we construct the trees by induction, then we prove that $T(\alpha, \beta)$ has order $\beta \cdot \alpha$. Finally we show that $T(\alpha, \beta)$ is isomorphic to a subtree of $T_{\beta \cdot \alpha}$ and hence is a minimal tree of order $\beta \cdot \alpha$ as required. The key to all of these proofs is to use induction on $\alpha$ for an arbitrary $\beta$.

**Definition 3.5. Replacement trees**

For each pair $\alpha, \beta < \omega_1$ we construct a tree $T(\alpha, \beta)$ and a map $f_{\alpha, \beta}: T(\alpha, \beta) \to T_\alpha$ satisfying:

(i) For each $x \in T_\alpha$ there exists $I = \{1\}$ or $\mathbb{N}$ and trees $t(x, j) \simeq T_\beta$, $j \in I$, such that

\[ f_{\alpha, \beta}^{-1}(x) = \bigcup_{j \in I} t(x, j) \] (incomparable union) with $I = \{1\}$ if $\alpha$ is a successor ordinal and $x$ is the unique initial node, or $\beta < \omega$, and $I = \mathbb{N}$ otherwise.

(ii) For each pair $a, b \in T(\alpha, \beta)$, $a \leq b$ implies $f_{\alpha, \beta}(a) \leq f_{\alpha, \beta}(b)$.

For each $\beta < \omega_1$, let $T(1, \beta) = T_\beta$ and $f_{1, \beta}: T(1, \beta) \to T_1$ be given by $f_{1, \beta}(a) = \{1\}$ $\forall a \in T(1, \beta)$.

Let $\alpha < \omega_1$ and suppose we have defined $T(\alpha, \beta)$ and $f_{\alpha, \beta}$ for each $\beta < \omega_1$. Roughly speaking, what we do to go from $\alpha$ to $\alpha + 1$ is to take $T_\beta$ and then after each of its terminal nodes we put a $T(\alpha, \beta)$ tree. This will give us the required tree, but we have to ensure that it is well defined and that we keep track of the order relation.

Recall from Definition 3.1 that $T_{\alpha+1} = \{a \cup \{\gamma + 1\} : a \in T_\alpha\} \cup \{\{\gamma + 1\}\}$ for some $\gamma < \omega_1$. Let $\delta_1, \delta_2$ be countable ordinals with $T(\alpha, \beta) \subset [1, \delta_1]^{<\omega}, T_\beta \subset [1, \delta_2]^{<\omega}$. Define a map $\tilde{\eta}: [1, \delta_2] \to [\delta_1 + 1, \delta_1 + \delta_2]$ by $\eta \mapsto \tilde{\eta} = \delta_1 + \eta$. For all ordinals $\lambda, \mu, \nu$, we have $\lambda + \mu = \lambda + \nu \Rightarrow \mu = \nu$, hence this map is one to one. Thus, if we define $\tilde{T}_\beta = \{\tilde{a} : a \in T_\beta\}$, then $\tilde{T}_\beta \simeq T_\beta$ as the map $\tilde{\eta}$ is merely relabeling the nodes, but the trees $\tilde{T}_\beta$ and $T(\alpha, \beta)$ are now incomparable since if $a \in \tilde{T}_\beta, b \in T(\alpha, \beta)$, then $a \cap b = \emptyset$. 

Let \((\tilde{x}_n)_{I}(I = \{1\} \text{ or } \mathbb{N})\) be the set of terminal nodes of \(\tilde{T}_\beta\), a sequence of incomparable nodes and let
\[
T(\alpha + 1, \beta) = \bigcup_{n \in I} \{a \cup \tilde{x}_n : a \in T(\alpha, \beta)\} \cup \tilde{T}_\beta
\]

\[
f_{\alpha + 1, \beta}(x) = \begin{cases} 
  f_{\alpha, \beta}(a) \cup \{\gamma + 1\} & x = a \cup \tilde{x}_n \ (a \in T(\alpha, \beta)) \\
  \{\gamma + 1\} & x \in \tilde{T}_\beta 
\end{cases}
\]

We need to show that the map \(f_{\alpha + 1, \beta}\) satisfies the required properties. Let \(y \in T_{\alpha + 1}\). If \(y = \{\gamma + 1\}\), then \(f_{\alpha + 1, \beta}^{-1}(y) = \tilde{T}_\beta \simeq T_\beta\). Otherwise \(y = a \cup \{\gamma + 1\}\) for some \(a \in T_\alpha\) and hence
\[
f_{\alpha + 1, \beta}^{-1}(y) = \bigcup_{n \in I} \{b \cup \tilde{x}_n : b \in f_{\alpha, \beta}^{-1}(a)\}
\]

\[
= \bigcup_{n \in I} \bigcup_{i \in I'} t_{n,i} \quad \text{where } t_{n,i} \simeq T_\beta \text{ and } I' = \{1\} \text{ or } \mathbb{N}
\]

\[
= \bigcup_{j \in I''} t(y, j) \quad \text{where } t(y, j) \simeq \tilde{T}_\beta \text{ and } I'' = \{1\} \text{ or } \mathbb{N}
\]
as required. Furthermore, the \(t(y, j)\)'s are incomparable. The second property is clear.

If \(\alpha\) is a limit ordinal, let \(\alpha_n \nearrow \alpha\) be the sequence of ordinals increasing to \(\alpha\) from Definition 3.1 and suppose we have constructed \(T(\alpha_n, \beta)\), \(f_{\alpha_n, \beta}\) for each \(\alpha_n\). Let \(T(\alpha_n, \beta) \subset [1, \delta_n)^{<\omega}\), \(\delta = \sup_n \delta_n < \omega_1\), and set \(\gamma_n = \delta + n\) for each \(n\). Then, as in the definition of the minimal trees, let \(\tilde{T}(\alpha_n, \beta) = \{a \cup \{\gamma_n\} : a \in T(\alpha_n, \beta)\}\), \(\tilde{f}_{\alpha_n, \beta}(a \cup \{\gamma_n\}) = f_{\alpha_n, \beta}(a)\), and let \(T(\alpha, \beta) = \bigcup_{n=1}^{\infty} \tilde{T}(\alpha_n, \beta)\), \(f_{\alpha, \beta} = \bigcup_{n=1}^{\infty} \tilde{f}_{\alpha_n, \beta}\).

**Lemma 3.6.** \(o(T(\alpha, \beta)) = \beta \cdot \alpha\).

**Proof.** We proceed by induction on \(\alpha\) for an arbitrary fixed \(\beta\). The result is obvious for \(\alpha = 1\).

Suppose \(o(T(\alpha, \beta)) = \beta \cdot \alpha\). By the construction of \(T(\alpha + 1, \beta)\) we have that \((T(\alpha + 1, \beta))^{\beta \cdot \alpha} = \tilde{T}_\beta\) and hence \(o(T(\alpha + 1, \beta)) = \beta \cdot \alpha + \beta = \beta \cdot (\alpha + 1)\). If \(\alpha\) is a limit ordinal and \(o(\tilde{T}(\alpha_n, \beta)) = o(T(\alpha_n, \beta)) = \beta \cdot \alpha_n\) for each \(n\), where \(T(\alpha, \beta) = \bigcup_{n=1}^{\infty} \tilde{T}(\alpha_n, \beta)\) from Definition 3.5, then \(o(T(\alpha, \beta)) = \sup_n o(\tilde{T}(\alpha_n, \beta)) = \sup_n \beta \cdot \alpha_n = \beta \cdot \alpha\).

The last of our results on these specially defined trees is the following:
Lemma 3.7. $T(\alpha, \beta)$ is a minimal tree of order $\beta \cdot \alpha$.

Proof. Since $o(T(\alpha, \beta)) = \beta \cdot \alpha$ and $T_{\beta \cdot \alpha}$ is a minimal tree of order $\beta \cdot \alpha$, then by Remark 3.4 it is sufficient to prove that $T(\alpha, \beta)$ is isomorphic to a subtree of $T_{\beta \cdot \alpha}$. We prove this by induction on $\alpha$ for an arbitrary $\beta$. The result is obvious for $\alpha = 1$ since $T(1, \beta) = T_\beta$.

Suppose $T(\alpha, \beta)$ is isomorphic to a subtree of $T_{\beta \cdot \alpha}$ and hence is a minimal tree of order $\beta \cdot \alpha$. Now, $o(T_{\beta \cdot (\alpha+1)}) = \beta \cdot (\alpha + 1)$ so $o((T_{\beta \cdot (\alpha+1)})^{\beta \cdot \alpha}) = \beta$, thus since $T_\beta$ is minimal it is isomorphic to a subtree of $(T_{\beta \cdot (\alpha+1)})^{\beta \cdot \alpha}$. But by construction $(T(\alpha+1, \beta))^{\beta \cdot \alpha} \simeq T_\beta$ and hence is isomorphic to a subtree $t_0$ of $(T_{\beta \cdot (\alpha+1)})^{\beta \cdot \alpha}$. Let the isomorphism which sends $(T(\alpha+1, \beta))^{\beta \cdot \alpha}$ onto $t_0 \subseteq (T_{\beta \cdot (\alpha+1)})^{\beta \cdot \alpha}$ be $a \mapsto a'$, so that if $(x_n)^\infty$ are the terminal nodes of $(T(\alpha+1, \beta))^{\beta \cdot \alpha}$, then $(x'_n)^\infty$ are their images in $(T_{\beta \cdot (\alpha+1)})^{\beta \cdot \alpha}$, the terminal nodes of $t_0$, under this map. Let $T(x'_n) = \{ y \in T_{\beta \cdot (\alpha+1)} : y > x'_n \} \subset T_{\beta \cdot (\alpha+1)}$, then $o(T(x'_n)) \geq \beta \cdot \alpha$ for each $n$. Now, by assumption, for each $n$ there exists a subtree $t_n$ of $T(x'_n)$ isomorphic to $T(\alpha, \beta)$ and hence the subtree $\tilde{T} = (\bigcup_1^\infty t_n) \cup t_0$ of $T_{\beta \cdot (\alpha+1)}$ is isomorphic to $T(\alpha+1, \beta)$ as required.

Let $\alpha$ be a limit ordinal with $T(\alpha, \beta) = \bigcup_1^\infty \tilde{T}(\alpha_n, \beta)$ via the construction in Definition 3.5, and let $T(\alpha_n, \beta)$ be isomorphic to a subtree of $T_{\beta \cdot \alpha_n}$ for each $n$. Then $\tilde{T}(\alpha_n, \beta)$ is isomorphic to a subtree of $\tilde{T}_{\beta \cdot \alpha_n}$ for all $n$, where $\tilde{T}_{\beta \cdot \alpha_n} = \{ a \cup \{ \gamma'_n \} : a \in T_{\beta \cdot \alpha_n} \}$ for some $\gamma'_n$, from Definition 3.1, and hence $T(\alpha, \beta)$ is isomorphic to a subtree of $T_{\beta \cdot \alpha} = \bigcup_1^\infty \tilde{T}_{\beta \cdot \alpha_n}$ as required. 

4. Proof of Theorem 1.1

We have shown everything we need about trees on subsets of ordinals and we now want to apply this to $\ell_1$-trees on a Banach space $X$.

Definition 4.1. Block of an $\ell_1$-tree

Let $T'$ be a subtree of an $\ell_1$-tree $T$. A block of $T'$ with respect to $T$ is a normalized vector $v$ in the linear span of some node $x = (x_1, \ldots, x_n) \in T'$ where either:

- $x$ is an initial node of $T$. 

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• the initial node of the subtree \( \{ y \in T' : y \leq x \} \) of \( T \) is an initial node of \( T \), or

• \((x_1, \ldots, x_m)\) is the predecessor node in \( T \) of the initial node of \( \{ y \in T' : y \leq x \} \) and \( v \) is in the linear span of \((x_{m+1}, \ldots, x_n)\).

If \( T' = T \), then a block of \( T \) is simply any normalized vector \( v \) in the linear span of any node \((x_1, \ldots, x_n)\) of \( T \).

**Lemma 4.2.** Let \( T \) be a tree on \( X \) of order \( \beta \cdot \alpha \) isomorphic to \( T(\alpha, \beta) \), and let \( F : T \rightarrow T_\alpha \) be the map from Definition 3.5 satisfying, for all \( x \in T_\alpha \), \( F^{-1}(x) = \bigcup_I T_n(x) \), where \( I = \{1\} \) or \( \mathbb{N} \), \( T_n(x) \simeq T_\beta \) and the \( T_n(x) \)'s are mutually incomparable. For each \( x \in T_\alpha \) and \( n \geq 1 \), let \( b(x, n) \) be a block of \( T_n(x) \) with respect to \( T \). Then there exists a block subtree \( T' \) of \( T \) and an isomorphism \( g : T' \sim T_\alpha \) satisfying: for every pair \( a, b \in T_\alpha \), with \( a < b \), there exist \( x_1 < \ldots < x_m \) in \( T_\alpha \), integers \( n_{x_1}, \ldots, n_{x_m} \) and \( k < m \) such that \( g^{-1}(a) = (b(x_i, n_{x_i}))_1^k \) and \( g^{-1}(b) = (b(x_i, n_{x_i}))_1^m \).

This sounds very complicated but all it is saying is that if you have a tree on \( X \), isomorphic to a replacement tree \( T(\alpha, \beta) \), then you can replace each \( \beta \)-subtree by a normalized vector in the linear span of a node of that tree, and refine to get a block subtree of order \( \alpha \).

**Proof.** As usual we prove this by induction on \( \alpha \) for an arbitrary \( \beta \). The result is obvious for \( \alpha = 1 \) and the only non-obvious case is the successor case.

Assume that the lemma is true for \( \alpha \). Let \( T \) be a tree on \( X \) of order \( \beta \cdot (\alpha + 1) \) isomorphic to \( T(\alpha + 1, \beta) \), let \( F : T \rightarrow T_{\alpha + 1} \) be the map with \( F^{-1}(x) = \bigcup_I T_n(x) \) where \( T_n(x) \simeq T_\beta \), and let \( b(x, n) \) be given for each \( x \in T_{\alpha + 1} \), \( n \in I \).

By construction of the replacement trees, \( T^{\beta \cdot \alpha} \simeq T_\beta \), and in fact \( T^{\beta \cdot \alpha} = F^{-1}(\{ \gamma + 1 \}) = T_1(\{ \gamma + 1 \}) \), where \( T_{\alpha + 1} = \{ a \cup \{ \gamma + 1 \} : a \in T_\alpha \} \cup \{ \{ \gamma + 1 \} \} \) from Definition 3.1. After each terminal node of \( T^{\beta \cdot \alpha} \) lies a tree isomorphic to \( T(\alpha, \beta) \). Let these trees be \( (t_j)_{j=1}^{\infty} \). Let \( j_0 \geq 1 \) be such that \( t_{j_0} > b(\{ \gamma + 1 \}, 1) \); then \( t_{j_0} \simeq T(\alpha, \beta) \) and so the lemma applies giving us a block subtree.
$t'_{j_0} \preceq t_{j_0}$ and $g : t'_{j_0} \rightarrow T_\alpha$ as in the statement. Now let

$$T' = \{(b(\{\gamma + 1\}, 1), u_1, \ldots, u_m) : (u_i)_{i=1}^m \in t'_{j_0}\} \cup \{(b(\{\gamma + 1\}, 1))\}$$

and let $G : T' \rightarrow T_{\alpha+1}$ by

$$G(a) = \begin{cases} 
  g((u_i)_{i=1}^m) \cup \{\gamma + 1\} & a = (b(\{\gamma + 1\}, 1), u_1, \ldots, u_m) \\
  \{\gamma + 1\} & a = (b(\{\gamma + 1\}, 1))
\end{cases}$$

then $G, T'$ clearly satisfy the lemma.

The proof where $\alpha$ is a limit ordinal just involves taking the union of the previous trees and functions.

\[\square\]

**Definition 4.3.** Restricted subtree of a tree.

Let $T$ be a tree on a set $X$ and let $T'$ be a subtree of $T$. We define another tree on $X$, the **restricted subtree** $R(T')$ of $T'$ with respect to $T$. Let $x = (x_i)_{i=1}^n \in T'$ and let $y$ be the unique initial node of $T'$ such that $y \leq x$; let $m \leq n$ be such that $y = (x_i)_{i=1}^m$. If $y$ is also an initial node of $T$, then set $k = 0$, otherwise let $k < m$ be such that $(x_i)_{i=1}^k$ is the predecessor node of $y$ in $T$. Finally, setting $R(x) = (x_{k+1}, \ldots, x_n)$, we define $R(T') = \{R(x) : x \in T'\}$. It is easy to see that $R(T')$ is isomorphic to $T'$.

**Proof of Theorem 1.1 for $p = 1$.** Let $T$ be an $\ell_1$-$K$-tree of order $\alpha^2$ on $X$. We show that there exists $T' \preceq T$ such that $T'$ is an $\ell_1$-$\sqrt{\kappa}$-tree of order $\alpha$.

By Lemmas 3.3 and 3.7, $T(\alpha, \alpha)$ is isomorphic to a subtree of $T$ and so we may assume that in fact $T(\alpha, \alpha)$ is isomorphic to $T$. Now let $F : T \rightarrow T_\alpha$ be the map from Definition 3.5 with $F^{-1}(x) = \bigcup_I T_n(x), T_n(x) \simeq T_\alpha$ for every $x \in T_\alpha$ and $n \in I$.

For each $x \in T_\alpha$ and $n \geq 1$ let $\tilde{T}_n(x) = R(T_n(x))$, the restriction being with respect to $T$. Note that $\tilde{T}_n(x)$ is an $\ell_1$-$K$-tree isomorphic to $T_n(x)$. If there exist $x \in T_\alpha$, $n \in I$ such that $\tilde{T}_n(x)$ is an $\ell_1$-$\sqrt{\kappa}$-tree we are finished, since $\tilde{T}_n(x)$ has order $\alpha$. Otherwise let $(x_1, \ldots, x_m)$ be a node of $\tilde{T}_n(x)$ which is not $\sqrt{\kappa}$ equivalent to the unit vector basis of $\ell_1^m$ and let $b(x, n) = \sum_{i=1}^m a_i x_i$ where
(a_i)_i \subset \mathbb{R}, \sum_{i=1}^m |a_i| > \sqrt{K} and \|b(x,n)\| = 1. Note that \|b(x,n)\| is a block of \( T_n(x) \) with respect to \( T \).

By Lemma 4.2 there exists \( T' \preceq T \) of order \( \alpha \) whose nodes are \( (b(x_1,n_{x_1}))_1^n \) for some \( n_{x_1} \), where \( \{x_1 < \cdots < x_m\} = \{y \in T : y \leq x\} \) for each \( x \in T_\alpha \). We need only show that this tree has constant \( \sqrt{K} \). Let \( (y_i)_1^n \) be a node in \( T' \) with parent node \( x = (x_1, \ldots, x_m) \in T \). Thus there exist subsets \( E_i \subset \{1, \ldots, m\}, E_1 < \cdots < E_n \) (where \( E < F \) means \( \max E < \min F \)) such that \( y_i = \sum_{k \in E_i} a_k x_k \) for each \( i \) and satisfying:

\[
1 = \|y_i\| = \left\| \sum_{E_i} a_k x_k \right\| < \frac{1}{\sqrt{K}} \sum_{E_i} |a_k|.
\]

Let \( (b_i)_1^n \subset \mathbb{R} \), then

\[
\left\| \sum_{i=1}^n b_i y_i \right\| = \left\| \sum_{i=1}^n b_i \sum_{k \in E_i} a_k x_k \right\|
\geq \frac{1}{K} \sum_{i=1}^n |b_i| \sum_{k \in E_i} |a_k|
> \frac{1}{K} \sum_{i=1}^n |b_i| \cdot \sqrt{K}
= \frac{1}{\sqrt{K}} \sum_{i=1}^n |b_i|
\]

as required. These last few lines are James’ argument.

Now, if we choose the smallest \( n \) so that \( K^{\frac{1}{2n}} \leq 1 + \varepsilon \), then we can iterate this argument to show that if \( T \) is an \( \ell_1-K \)-tree of order \( \alpha^{2^n} \), then there exists \( T' \preceq T \) such that \( T' \) is an \( \ell_1-(1 + \varepsilon) \)-tree of order \( \alpha \), which proves the theorem.

\[\square\]

Remark 4.4.

(i) The proof of Theorem 1.1 for \( p = \infty \) is very similar to that for \( p = 1 \), except that given an \( \ell_\infty-K \)-tree \( T \) on \( X \) of order \( \alpha^{2^n} \) for \( n \) sufficiently large, we choose a block subtree \( T' \preceq T \) of order \( \alpha \) to obtain \( \| \sum_1^n a_i x_i \| \leq (1 + \varepsilon) \sup_i |a_i| \) for all nodes \( (x_i)_1^n \in T' \), and then the lower estimate follows automatically according to [J].
(ii) The proof of the theorem also gives some fixed points—that is, ordinals $\alpha$ such that if we have an $\ell_1$-$K$-tree of order $\alpha$, then for any $\varepsilon > 0$ we can get a block subtree of this which is an $\ell_1$-$(1 + \varepsilon)$-tree also of order $\alpha$. In fact we see from the proof that this is true for every countable ordinal $\alpha$ which satisfies $\beta < \alpha$ implies $\beta^n < \alpha$ for each $n \geq 1$. From basic results on ordinals we see that $\alpha$ satisfies this condition if and only if $\alpha$ is of the form $\alpha = \omega^\gamma$ for some ordinal $\gamma$ (see Fact 5.3 below).

5. Calculating the $\ell_1$ index of a Banach space

**Definition 5.1.** Block basis tree

A block basis tree on a Banach space $X$, with respect to a basis $\langle e_i \rangle_{i=1}^{\infty}$ for $X$, is a tree $T$ on $X$ such that every node $(x_i)_{i=1}^{n}$ of $T$ is a block basis of $\langle e_i \rangle_{i=1}^{\infty}$. Moreover, if $T$ is also an $\ell_1$-$K$-tree, then we say $T$ is an $\ell_1$-$K$-block basis tree.

**Definition 5.2.** The $\ell_1$-indices of a Banach space $X$: $I(X)$ and $I_b(X)$.

Let $X$ be a separable Banach space and for each $K \geq 1$ set

$$I(X, K) = \sup \{ o(T) : T \text{ is an } \ell_1$-$K$-tree on } X \} .$$

The **Bourgain $\ell_1$-index** of $X$ [B] is then given by

$$I(X) = \sup_{1 \leq K < \infty} \{ I(X, K) \} .$$

By Bourgain, $I(X) < \omega_1$ if and only if $X$ does not contain $\ell_1$.

The block basis index is the analogous index to $I(X)$ except that it is only defined on block basis trees. For a Banach space $X$ with a basis $\langle e_i \rangle$, and $K \geq 1$, set

$$I_b(X, K, (e_i)) = \sup \{ o(T) : T \text{ is an } \ell_1$-$K$-block basis tree w.r.t. $(e_i)$ on } X \} .$$

The **block basis index** is then given by

$$I_b(X, (e_i)) = \sup \{ I_b(X, K, (e_i)) : 1 \leq K < \infty \} .$$
When the basis in question is fixed we shall write $I_b(X,K)$ rather than $I_b(X,K,(e_i))$ etc. It is worth recalling here that $I_b(X)$ is not in general independent of the basis. It is clear, however, that $I_b(X,K,(e_i)) \leq I(X,K)$ for every $X$, $K$ and $(e_i)$.

We next state some facts about ordinals. The proofs may be found in Monk [M].

**Fact 5.3.** Let $\alpha$ be an infinite countable ordinal. Then the following statements hold:

(i) There exist $k \geq 1$, (countable) ordinals $\theta_1 > \cdots > \theta_k \geq 0$ and $n_i \geq 1$ ($i = 1, \ldots, k$), uniquely determined by $\alpha$ such that $\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k$. This is the Cantor normal form of an ordinal.

(ii) For all $\beta < \alpha$, $\beta \cdot 2 < \alpha$ if and only if there exists $\gamma < \omega_1$ such that $\alpha = \omega^\gamma$.

(iii) For all $\beta < \alpha$, $\beta^2 < \alpha$ if and only if there exists $\gamma < \omega_1$ such that $\alpha = \omega^{\omega\gamma}$.

(iv) If $\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k$, then $\omega \cdot \alpha = \alpha$ if and only if $\theta_k \geq \omega$.

(v) If $\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k$, then $\alpha \cdot \omega = \omega^{\theta_1+1}$.

Our first result of this section is to show how we may refine $\ell_1$-trees in a Banach space with a basis to get $\ell_1$-block basis trees, and explain how this relates to the indices. We then show that both $I(X)$ and $I_b(X)$ are of the form $\omega^\alpha$ for some $\alpha$, and that if $\alpha \geq \omega$ for either index, then the indices are the same. The block basis trees are much easier to work with, and once we have the block index of a space we have a good idea what the index is. In the second part of this section we use this idea to calculate the index of some Tsirelson type spaces.

**Notation**

For a Banach space $X$ let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ denote the unit ball and unit sphere of $X$ respectively. If $(x_i)_{i \in I} \subset X$, where $I \subset \mathbb{N}$, let $[x_i]_{i \in I}$ be the closed linear span of these vectors.

If $X$ is a Banach space with basis $(e_i)_{i=1}^\infty$, let $E_n = [e_i]_{i=1}^n$, let $P_n : X \to E_n$ be the basis projection onto $E_n$ given by $P_n(\sum a_ie_i) = \sum_{i=1}^n a_ie_i$, and let $X_n = [e_i]_{i=n+1}^\infty$. Finally, we define the support of
$x \in X$ with respect to $(e_i)_i^{\infty}$ as $\text{supp}(x) = \{ n \geq 1 : (P_n - P_{n-1})(x) \neq 0 \}$. Thus, if $x = \sum_{i \in F} a_i e_i$ with $a_i \neq 0$ for $i \in F$, then $\text{supp}(x) = F$. If $x = (x_1, \ldots, x_n)$ is a sequence of vectors, then $\text{supp}(x) = \bigcup_{i=1}^n \text{supp}(x_i)$. In the following $X$ will always denote a separable Banach space not containing $\ell_1$.

**Proposition 5.4.** Let $X$ have a basis, then $I(X, K) \geq \omega \cdot \alpha$ implies that $I_b(X, K + \varepsilon) \geq \alpha$ for every $\varepsilon > 0$.

We first prove the following elementary lemma:

**Lemma 5.5.** Let $X$ be a Banach space with basis $(e_i)_i^{\infty}$ and let $T$ be an $\ell_1$-tree of order $\omega$ on $X$, then for each $n \geq 1$ there exists a block $x$ of $T$ with $P_n x = 0$.

**Proof.** There exists $m > n$ such that the linear space spanned by $(y_i)_i^{m} \in T$ has dimension greater than $n$. Thus the restriction of $P_n$ to $[y_i]_1^{m}$ is not one to one and hence there exists $x \in [y_i]_1^{m}$ with $\|x\| = 1$ and $P_n x = 0$. \hfill $\Box$

**Proof of Proposition 5.4.** If $I(X, K) \geq \omega \cdot \alpha$, then there exists an $\ell_1$-$K$-tree $T$ on $X$ of order $\omega \cdot \alpha$ and this in turn, by Lemma 3.7, has an $\ell_1$-$K$-subtree $T'$ isomorphic to $T(\alpha, \omega)$. Thus we may assume that $T$ itself is isomorphic to $T(\alpha, \omega)$. We prove the following statement:

For all $\alpha < \omega_1$, each $l \geq 0$, and every $\varepsilon > 0$, if $T$ is an $\ell_1$-$K$-tree isomorphic to $T(\alpha, \omega)$, then there exists an $\ell_1$-$K$-block subtree $T'$ of $T$ of order $\alpha$ such that for any node $(y_i)_i^{m} \in T'$ there exists $l = k(1) < \cdots < k(m + 1)$ with $\|y_i - P_{k(i+1)}y_i\| < \varepsilon$ and $P_{k(i)}y_i = 0$ ($i = 1, \ldots, m$).

We induct on $\alpha$; the statement is clear for $\alpha = 1$ by Lemma 5.5. Suppose we have proved the statement for $\alpha$, and let $T \simeq T(\alpha + 1, \omega)$. Let $F : T \rightarrow T_{\alpha+1}$ be the map $F^{-1}(x) = \bigcup_{I} T_n(x)$ ($I = \{1\}$ or $\mathbb{N}$), from Definition 3.5, such that $T_n(x) \simeq T_\omega$ for each $n$ and $x$, and the $T_n(x)$’s are mutually incomparable. Let $z$ be the unique initial node of $T_{\alpha+1}$. By Lemma 5.5 we can find a block $b(1, z)$ of $T_1(z)$ such that $P_l b(1, z) = 0$ and we can find $l' > l$ such that $\|b(1, z) - P_{l'} b(1, z)\| < \varepsilon$. Let $\tilde{T}$
be a subtree of $T$ isomorphic to $T(\alpha, \omega)$ with $b(1, z) < \tilde{T}$. Applying the induction hypothesis to $\tilde{T}$ we obtain $\tilde{T}' \preceq \tilde{T}$ such that $P_i y_i = 0$ for every node $(y_i)_{1}^{m} \in \tilde{T}'$. Let $T' = \{(b(1, z), y_1, \ldots, y_m) : (y_i)_{1}^{m} \in \tilde{T}' \} \cup \{(b(1, z))\}$. Then $T'$ is the required block subtree.

Now let $\alpha$ be a limit ordinal and suppose we have proved the statement for each $\beta < \alpha$. Let $(\alpha_n)$ be the sequence of ordinals increasing to $\alpha$ such that $T = \bigcup_{n=1}^{\infty} T(n)$ where the trees $T(n)$ are mutually incomparable and $T(n) \simeq T(\alpha_n, \omega)$. Applying the hypothesis to each $T(n)$ we obtain block subtrees $T(n)' \preceq T(n)$. Then $T' = \bigcup_{n=1}^{\infty} T(n)'$ is the required block subtree.

Thus, if we have an $\ell_1$-K-tree $T$ of order $\omega \cdot \alpha$ and $\varepsilon' > 0$, then let $T'$ be the $\ell_1$-K-block subtree of $T$ from above. For each node $(y_i)_{1}^{m}$ of $T'$ let $(k(i))_{1}^{m+1} \subset \mathbb{N}$ be the sequence from above and let

$$v_i = \frac{P_{k(i+1)} y_i}{\|P_{k(i+1)} y_i\|} \quad (i = 1, \ldots, m).$$

The sequence $(v_i)_{1}^{m}$ is a uniform perturbation of a basis $K$ equivalent to the unit vector basis of $\ell_1^m$. Hence, if $\varepsilon'$ is chosen sufficiently small, then $(v_i)_{1}^{m}$ is $K + \varepsilon$ equivalent to the unit vector basis of $\ell_1^m$.

This completes the proof since if we replace the nodes $(y_i)_{1}^{m}$ with $(v_i)_{1}^{m}$ as above, then we obtain $\tilde{T}$, a block basis tree of order $\alpha$ and constant $(K + \varepsilon)$, so that $I_b(X, K + \varepsilon) \geq \alpha$ as required. \qed

**Theorem 5.6.** Let $X$ be a Banach space with a basis, then $I_b(X) = \omega^\alpha$ for some $\alpha < \omega_1$.

To prove this theorem we need some preliminary results. We first show that there is no $\ell_1$-K-block basis tree whose order is the same as the block basis index, and hence $I_b(X)$ must be a limit ordinal. Then we show that $\beta < I_b(X)$ implies that $\beta \cdot 2 < I_b(X)$, which completes the proof.

**Lemma 5.7.** Let $X$ be a Banach space with a basis and $K \geq 1$, then $I_b(X, K) \neq I_b(X)$. In particular $I_b(X)$ is a limit ordinal.

**Proof.** We prove by induction on $\alpha$ that for every Banach space $X$ with a basis and any $K \geq 1$, if $I_b(X, K) = \alpha$, then $I_b(X) > \alpha$. This is trivial for $\alpha = 1$.

Let the result be true for $\alpha$ and suppose, if possible, that it is false for $\alpha + 1$. Let $X$ be a Banach space with basis $(e_i)_{1}^{\infty}$ and $K \geq 1$ such that $I_b(X, K) = I_b(X) = \alpha + 1$. Now there exists an
\(\ell_1\)-\(K\)-block basis tree \(T\) of order \(\alpha + 1\) isomorphic to the minimal tree \(T_{\alpha+1}\). Let \(x = (x_1)\) be the unique initial node of \(T\), let \(k = \max(\text{supp} \ x_1)\), let \(X_k\) be the subspace of \(X\) spanned by \((e_i)_{i > k}\) and let \(T(\alpha) = \{(y_i)^n_1 : y = (x_1, y_1, \ldots, y_m) \in T\) and \(y > x\}\). Clearly \(T(\alpha)\) is an \(\ell_1\)-\(K\)-block basis tree on \(X_k\) of order \(\alpha\), and so \(I_b(X_k) > \alpha\), otherwise \(I_b(X_k, K) = \alpha = I_b(X_k)\) contradicting our assumption. Thus there exists an \(\ell_1\)-block basis tree \(T'\) on \(X_k\) of order \(\alpha + 1\) for some constant \(K' \geq 1\). But now the tree \(\tilde{T} = \{(x_1, u_1, \ldots, u_i) : (u_1, \ldots, u_i) \in T'\} \cup \{(x_1)\}\) is an \(\ell_1\)-block basis tree on \(X\) of order \(\alpha + 2\) for some constant \(K''\) contradicting the assumption that \(I_b(X) = \alpha + 1\). This proves the result for \(\alpha + 1\).

Let \(\alpha\) be a limit ordinal and suppose the result is true for every \(\alpha' < \alpha\), but false for \(\alpha\). Again let \(X\) be a Banach space with basis \((e_i)_{\alpha}^\infty\), \(K \geq 1\) such that \(I_b(X, K) = I_b(X) = \alpha\) and \(T\) an \(\ell_1\)-\(K\)-block basis tree of order \(\alpha\) isomorphic to the minimal tree \(T_{\alpha}\). By Lemma 3.2 there exists a sequence of ordinals \((\alpha_n)\) such that \(\alpha = \sup_n (\alpha_n + 1) = \sup_n \alpha_n\) and mutually incomparable trees \(t_n\) for each \(n\) such that \(t_n \simeq T_{\alpha_n+1}\) and \(T = \cup_n t_n\). For each \(n\) let \(z_n = (w_i)^{\alpha_n}_{1}\) be the unique initial node of \(t_n\) and let \(t_n' = \{(y_i)^n_1 : y = (w_1, \ldots, w_{\alpha_n}, y_1, \ldots, y_m) \in t_n\) and \(y > z_n\}\), a tree isomorphic to \(T_{\alpha_n}\). Clearly \(T' = \cup_n t_n'\) is a tree on \(X_1\) with order \(\alpha\). Let \(\tilde{T} = \{(e_1, u_1, \ldots, u_i) : (u_1, \ldots, u_i) \in T'\} \cup \{(e_1)\}\). This is an \(\ell_1\)-block basis tree of order \(\alpha + 1\), contradicting the assumption that \(I_b(X) = \alpha\). This proves the first part of the lemma.

Suppose, if possible, that \(I_b(X) = \alpha + 1\) for some \(\alpha\). Then there exists an \(\ell_1\)-\(K\)-block basis tree \(T\) of order \(\alpha + 1\) for some \(K\) contradicting the previous result. \[\square\]

**Lemma 5.8.** Let \(X\) be a Banach space with basis \((e_i)_{\alpha}^\infty\). If \(\beta < I_b(X)\), then there exists \(K > 1\) such that \(I_b(X_n, K) \geq \beta\) for every \(n \geq 1\).

**Proof.** The result is trivial for \(\beta < \omega\). Suppose first that \(\beta < I_b(X)\) is a limit ordinal and let \(T\) be an \(\ell_1\)-\(K\)-block basis tree on \(X\) of order \(\beta\). Let \(T(n) = \{(x_i)^{l}_{n+1} : \exists(x_i) \in T\) with \(l > n\}\). \(T(n)\) is clearly a block subtree of \(T\) and an \(\ell_1\)-\(K\)-block basis tree on \(X_n\). Moreover, \(o(T(n)) = \beta\), otherwise \(o(T) \leq o(T(n)) + n < \beta\), a contradiction.
Now let $\beta < I_b(X)$ be a successor ordinal greater than $\omega$, then $\beta = \beta' + k$ for some limit ordinal $\beta' \geq \omega$ and $k \geq 1$. From the limit ordinal case there exists $K > 1$ such that $I_b(X_m, K) \geq \beta'$ for every $m \geq 1$. Now, $X$ contains $\ell^m_1$'s uniformly so there exists $m > n$ and a normalized block basis $(x_i)_1^m$ of $[e_i]_{m+1}$ which is $K$ equivalent to the unit vector basis of $\ell^m_1$. Let $T$ be an $\ell_1$-$K$-block basis tree on $X_m$ of order $\beta'$ and let $T(n) = \{(x_1, \ldots, x_k, u_1, \ldots, u_l) : (u_1, \ldots, u_l) \in T \}$, then $T(n)$ is an $\ell_1$-block basis tree on $X_n$ of order $\beta' + k = \beta$ and some constant which depends only on $K$. □

**Proof of Theorem 5.6.** We show that if $\beta < I_b(X)$, then $\beta \cdot 2 < I_b(X)$, which is enough to prove the theorem by Fact 5.3 (ii). Let $\beta < I_b(X)$ and $T$ be an $\ell_1$-$K$-block basis tree on $X$ of order $\beta$. For each $n$ let $T(n)$ be an $\ell_1$-$K$-block basis tree on $X_n$ of order $\beta$ from Lemma 5.8. Let $(a_i)$ be the collection of terminal nodes of $T$ and for each $i \geq 1$ let $n(i) = \max(\text{supp} a_i)$. Finally, setting $\tilde{T}(n(i)) = \{a_i \cup x : x \in T(n(i))\}$, we have that $\tilde{T} = T \cup (\cup_i \tilde{T}(n(i)))$ is an $\ell_1$-block basis tree of order $\beta \cdot 2$ and hence $I_b(X) > \beta \cdot 2$ as required. □

**Theorem 5.9.** Let $X$ be a separable Banach space, then $I(X) = \omega^\alpha$ for some $\alpha < \omega_1$.

The proof of this theorem is similar to that of Theorem 5.6, but without a basis for $X$ we have to work harder.

**Lemma 5.10.** Let $T$ be a countable tree of order $\alpha < \omega_1$, $M$ the collection of maximal nodes of $T$, $M = \cup_{i=1}^n M_i$ a partition of $M$, and $T_i = \{x \in T : x \leq m \text{ for some } m \in M_i\}$. Then $o(T_i) = \alpha$ for some $1 \leq i \leq n$.

**Proof.** We prove by induction on $\alpha$. The result is obvious for $\alpha = 1$. Suppose it is true for $\alpha$, and let $T$ be a countable tree of order $\alpha + 1$ with $M$, $M_i$, $T_i$ as above. Let $(a_j)$ be the sequence of initial nodes of $T$ and $t_j = \{x \in T : x \geq a_j\}$. Clearly the $t_j$'s are mutually incomparable and $T = \cup_j t_j$, hence $o(t_{j_0}) = \alpha + 1$ for some $j_0$. Let $t' = \{x \in T : x > a_{j_0}\}$, then $o(t') = \alpha$. Now, $M = \cup_{i=1}^n M_i$ also partitions the terminal nodes of $t'$ and setting $t'_i = \{x \in t' : x \leq m \text{ for some } m \in M_i\}$ we have
\( o(t'_{i_0}) = \alpha \) for some \( i_0 \) by assumption. Now \( \{a_{j_0}\} \cup t'_{i_0} \) is a tree of order \( \alpha + 1 \) and \( \{a_{j_0}\} \cup t'_{i_0} \subseteq T_{i_0} \). Thus \( o(T_{i_0}) = \alpha + 1 \) as required.

Let \( \alpha \) be a limit ordinal and suppose the result is true for each \( \alpha' < \alpha \). Write \( T = \bigcup t_k \) as a union of mutually incomparable trees \( t_k \) of order \( \alpha_k \) where \( \text{sup} \alpha_k = \alpha \). Given \( M; M_i; T_i \) as above let \( t_{k,i} = \{x \in t_k : x \leq m \text{ for some } m \in M_i\} \) and let \( i(k) \in \{1, \ldots, n\} \) satisfy \( o(t_{k,i(k)}) = \alpha_k \) for each \( k \), by assumption. Let \( N_i = \{k \geq 1 : i(k) = i\} \), then \( N_i \) must be infinite for some \( i_0 \), so let \( N_{i_0} = (k_j)_{j}^{\infty} \). Now for each \( j \) we have \( t_{k_j,i(k_j)} = t_{k_j,i_0} \subseteq T_{i_0} \) and the trees \( t_{k_j,i_0} \) are mutually incomparable, thus \( o(\bigcup t_{k_j,i_0}) = \alpha \) which implies \( o(T_{i_0}) = \alpha \) as required. \( \square \)

**Lemma 5.11.** Let \( X \) be a separable Banach space not containing \( \ell_1 \) and \( K \geq 1 \), then \( I(X,K) \neq I(X) \). In particular \( I(X) \) is a limit ordinal.

**Proof.** Let \( I(X,K) = \alpha \) for some \( \alpha < \omega_1 \) and \( T \) be an \( \ell_1-K \)-tree on \( X \) of order \( \alpha \). Recall that a Banach space \( X \) is \( \mathcal{L}_1-K \) if there exists a collection \( (E_n) \) of finite dimensional subspaces of \( X \) with \( d(E_n, \ell_1^{\dim E_n}) \leq K \) for every \( n \), and for each finite set \( F \subseteq X \) and all \( \varepsilon > 0 \) there exists \( n \) such that the distance from \( x \) to \( E_n \) is less than \( \varepsilon \) for all \( x \) in \( F \). Also recall that every infinite dimensional \( \mathcal{L}_1 \) space contains \( \ell_1 \). See [LT] for more information on \( \mathcal{L}_1 \) spaces.

Now let \( M \) be the set of maximal nodes of \( T \). Clearly this defines a collection of finite dimensional subspaces \( [x_i]_1^n \) such that \( d([x_i]_1^n, \ell_1^n) \leq K \), where \( (x_i)_1^n \in M \). Thus, since \( X \) doesn’t contain \( \ell_1 \), it is not a \( \mathcal{L}_1 \) space and hence there exist \( F = \{z_1, \ldots, z_r\} \subseteq S(X) \) and \( \varepsilon > 0 \) such that for each \( m = (x_i)_1^n \in M \) there exists \( i(m) \in \{1, \ldots, r\} \) with \( d(z_{i(m)}, S([x_i]_1^n)) > \varepsilon \). For \( i = 1, \ldots, r \) set \( M_i = \{m \in M : i(m) = i\} \). Then \( M = \bigcup_i M_i \) partitions \( M \) and defines \( T = \bigcup_i T_i \) as in Lemma 5.10. So, from the lemma, we have \( o(T_{i_0}) = \alpha \) for some \( i_0 \leq r \). Let \( T' = \{(z_{i_0}, u_1, \ldots, u_m) : (u_1, \ldots, u_m) \in T_{i_0} \} \cup \{(z_{i_0})\} \), then this is an \( \ell_1 \)-tree on \( X \), for some constant \( K' = K'(K, \varepsilon) \), of order \( \alpha + 1 \). Thus \( I(X) > \alpha \) so \( I(X,K) \) which completes the first part of the proof. The argument that \( I(X) \) is a limit ordinal is the same as for \( I_0(X) \). \( \square \)
Lemma 5.12. Let $T$ be a tree on $X$ of order $\alpha$, where $\alpha$ is a limit ordinal. Let $F \subset S(X^*)$ be finite and $X_F = \{x \in X : x^*(x) = 0 \ \forall x^* \in F\}$. Then there exists a block subtree $T'$ of $T$ with $o(T') = \alpha$ and $T' \subseteq X_F$.

Proof. Let $|F| = n$. We note that $\alpha$ is a limit ordinal if and only if $\alpha = \omega \cdot \beta$ for some ordinal $\beta$, and prove the lemma by induction on $\beta$.

For $\beta = 1$, $\alpha = \omega$, and let $T$ be isomorphic to $T_\omega$. Notice that if $(x_i)_{i=1}^{n+1} \in T$, then there exists $x \in S([x_i]_{i=1}^{n+1})$ with $x \in X_F$. Thus for each $k$ there exists a node $(x_i^k)_{i=1}^l \in T$, for $l$ sufficiently large, from which we may extract a normalized block basis $(y_j^k)_{j=1}^k \subseteq X_F$ and such that $T' = \{(y_j^1, \ldots, y_j^k) : 1 \leq j \leq k, k \geq 1\}$ is a block subtree of $T$. This is now the required tree.

Suppose the result is true for $\beta$ and let $\alpha = \omega \cdot (\beta + 1) = \omega \cdot \beta + \omega$ and $T$ be a tree of order $\alpha$. Since $T$ has a subtree isomorphic to $T_\alpha$ we may assume $T \simeq T_\alpha$. Now $T^\omega \cdot \beta$ is isomorphic to $T_\omega$ and we apply the case $\beta = 1$ to obtain a block subtree $\tilde{T} \subset T^\omega \cdot \beta$ of order $\omega$, contained in $X_F$. Let $(\tilde{a}_i)$ be the sequence of terminal nodes in $\tilde{T}$ and $a_i$ the parent node of $\tilde{a}_i$ in $T^\omega \cdot \beta$ for each $i$. Let $T(i) = \{x \in T : x > a_i\}$, then $o(T(i)) \geq \omega \cdot \beta$. Thus we may apply the induction hypothesis to $R(T(i))$ (the restricted tree from Definition 4.3) for each $i$ to obtain block subtrees $T(i)' \subset T(i)$ with $o(T(i)') = \omega \cdot \beta$ and $T(i)' \subset X_F$. Finally, $T' = \tilde{T} \cup (\cup_i T(i)')$ is the required tree of order $\alpha$.

Let $\beta$ be a limit ordinal and suppose the result is true for all $\beta' < \beta$. Let $(\beta_n)$ be the increasing sequence of ordinals whose limit is $\beta$, then $\alpha = \omega \cdot \beta = \sup_n \omega \cdot \beta_n$ so that if $T$ is a tree of order $\alpha$ and then $T$ contains mutually incomparable trees of order $\omega \cdot \beta_n$ for each $\beta_n$. We apply the hypothesis to each of these trees to obtain the result. □

Proof of Theorem 5.9. By Lemma 5.11, $I(X) = \alpha$ for some limit ordinal $\alpha$. We show that if $\beta < \alpha$ is a limit ordinal, then $\beta \cdot 2 < \alpha$. It follows that if $\beta < \alpha$ is a successor ordinal, then $\beta \cdot 2 < \alpha$. This is enough to prove the theorem by Fact 5.3.
Let $T$ be an $\ell_1$-$K$-tree of order $\beta$ for some $K$. If $(x_i)_{i1}^n \in T$ let $F = F((x_i)_{i1}^n) \subset S(X^*)$ be a finite set which 1-norms a $(1/2)$-net in $S([x_i]_{i1}^n)$. Choose by Lemma 5.12 $T_{(x_i)_{i1}^n,F}$ a block subtree of $T$ of order $\beta$ contained in $X_F$. Let $(a_k)$ be the collection of maximal nodes of $T$ and if $a_k = (x_i)_{i1}^n$ let $T(k) = \{a_k \cup x : x \in T_{(x_i)_{i1}^n,F}\}$. Thus $T' = T \cup (\cup_k T(k))$ is an $\ell_1$-$6K$-tree of order $\beta \cdot 2$ as required, and hence $\beta \cdot 2 < \alpha$. \hfill \qed

**Corollary 5.13.** Let $X$ have a basis. If $I(X) \geq \omega^{\omega}$, then $I(X) = I_b(X)$.  

**Proof.** Let $\alpha > \omega$, and suppose $I(X) = \omega^{\alpha}$. Then for every $\beta$ with $\omega \leq \beta < \alpha$ there exists $K$ such that $I(X,K) \geq \omega \cdot \omega^\beta = \omega^\beta$. Thus $I_b(X,K') \geq \omega^\beta$ for some $K'$ by Proposition 5.4, and hence $I_b(X) > \omega^\beta$ for every $\beta < \alpha$. If $\alpha$ is a limit ordinal, then $\omega^\alpha = \sup_{\beta < \alpha} \omega^\beta$ and so $I_b(X) \geq \omega^\alpha = I(X)$. Otherwise $\alpha = \alpha' + 1$, where $\alpha' \geq \omega$ and $I_b(X) > \omega^{\alpha'}$, which implies $I_b(X) \geq \omega^{\alpha'} = I(X)$ since $I_b(X) = \omega^{\gamma}$ for some $\gamma$ by Theorem 5.6. In either case we know that $I(X) \geq I_b(X)$ and so they are equal.

If $I(X) = \omega^{\omega}$, then $I(X) > \omega^{n+1}$ for every $n \geq 1$ and hence $I_b(X) > \omega^n$ for every $n \geq 1$ by Proposition 5.4. Thus $I_b(X) \geq \omega^\omega = I(X)$ and so $I(X) = I_b(X)$ as required. \hfill \qed

**Corollary 5.14.** If $I(X) = \omega^n$, then $I_b(X) = \omega^m$ where $m = n$ or $n - 1$.  

**Proof.** This follows from similar arguments to those for the previous corollary. \hfill \qed

**Remark 5.15.** We collect together some notes about which values $\alpha$ may take when $I(X) = \omega^\alpha$.

(i) If $X$ does not contain $\ell_1^n$’s uniformly, then $I(X) = \omega = I_b(X)$. Also, if $X$ contains $\ell_1^n$’s uniformly, then $I(X) \geq \omega^2$. It is easy to see that $I_b(c_0) = \omega$ (where the block basis index is calculated with respect to the unit vector basis for $c_0$) and so $I(c_0) = \omega^2$ by Corollary 5.14. Thus the two ordinal indices may indeed differ. In fact, by Remark 5.21 below, for each $n \geq 1$ there exists a Banach space $X_n$ with $I_b(X_n) = I(X_n) = \omega^{n+1}$ and for each $n \geq 1$ there exists a Banach space $Y_n$ with $I_b(Y_n) = \omega^n$ while $I(Y_n) = \omega^{n+1}$.
(ii) If \( I(X) < \omega^\alpha \), then it is possible for a space \( X \) to have two bases \( (x_i) \) and \( (y_i) \) with \( I_b(X, (x_i)) \neq I_b(X, (y_i)) \). Indeed, for each \( n \geq 1 \) let \( H_n \) be the span of the first \( 2^n \) Haar functions in \( C(\Delta) \) (where \( \Delta \) is the Cantor set on \([0, 1] \)); if \( X = (\sum H_n)_{c_0} \), then \( X \simeq c_0 \). Thus, if \( (x_i) \) is a basis for \( X \) equivalent to the unit vector basis of \( c_0 \), then \( I_b(X, (x_i)) = \omega \). However, if \( (y_i) \) is the basis for \( X \) consisting of the Haar bases for the \( H_n \)'s strung together, then, since each basis for \( H_n \) admits a block basis of length \( n \) which is 1-equivalent to the unit vector basis of \( \ell_1^n \), we obtain \( I_b(X, (y_i)) = \omega^2 \). By Corollary 5.14 the block basis indices for different bases can only differ by a factor of \( \omega \).

(iii) We note here that there are some ordinals \( \alpha \) for which there are no spaces \( X \) with index \( I(X) = \omega^\alpha \). In particular, if \( \alpha \) is a limit ordinal, then there is no space \( X \) with \( I(X) = \omega^{\omega^\alpha} \). Otherwise, let \( I(X) = \omega^\omega \), then for all \( \alpha' < \alpha \) there is some \( K \) such that there exists an \( \ell_1-K \)-tree of order \( \omega^{\omega^\alpha'} \), which we may then refine to get an \( \ell_1-(1+\varepsilon) \)-block subtree of order \( \omega^{\omega^\alpha'} \) for any \( \varepsilon > 0 \), by Remark 4.4 (ii). Hence \( X \) contains a block basis tree of constant 2 and order \( \omega^\omega \) (taking the union of these trees) and so \( I(X) \geq \omega^{\omega^\alpha+1} \).

(iv) By Remark 5.21 below, for every \( \alpha < \omega_1 \) there exists a Banach space \( X \) with \( I(X) = \omega^{\alpha+1} \), and by Theorem 5.19 below, there exists a Banach space \( Y = T(S_{\omega^\alpha}, 1/2) \) with \( I(Y) = \omega^{\omega^{\alpha+1}} \).

(v) If \( X \) is asymptotic \( \ell_1 \) (see for example [OTW] for the definition of this), then \( I_b(X) \geq \omega^\omega \) and so \( I(X) = I_b(X) \).

**Question 1.** For which limit ordinals \( \alpha \) do there exist Banach spaces \( X \) with index \( I(X) = \omega^\alpha \)?

We have already shown that there exist Banach spaces with index \( \omega^{\alpha+1} \) for every \( \alpha < \omega_1 \). We have also shown that we cannot have indices of the form \( \omega^{\omega^\alpha} \) for \( \alpha \) a limit ordinal, and that we do have spaces with index of the form \( \omega^{\omega^{\alpha+1}} \), but this leaves the question open for all other limit ordinals.
This completes the first part of the section. We now apply some of these results and methods to calculating the $\ell_1$ index of some Tsirelson spaces.

**Definition 5.16.** Schreier sets of order $\alpha$, $S_\alpha$ [AA].

Let $E, F \subseteq \mathbb{N}, n \geq 1$. We write $E < F$ if $\max E < \min F$ and $n < E$ if $\{n\} < E$. Let $\mathcal{M}, \mathcal{N}$ be collections of finite sets of integers and $K = (k_i) \subseteq \mathbb{N}$. We define

$$\mathcal{M}[\mathcal{N}] = \{\bigcup_{i=1}^{k} F_i : F_i \in \mathcal{N} \ (i = 1, \ldots, k) \} \text{ and } \exists E = \{m_1, \ldots, m_k\} \in \mathcal{M}$$

with $m_1 \leq F_1 < m_2 \leq F_2 < \cdots < m_k \leq F_k; \ k \geq 1$.

and $\mathcal{M}(K) = \{\{k_i : i \in E\} : E \in \mathcal{M}\}$.

The Schreier sets, $S_\alpha$ for each $\alpha < \omega_1$ are defined inductively as follows: Let $S_0 = \{\{n\} : n \geq 1\} \cup \{\emptyset\}$, $S_1 = \{F \subset \mathbb{N} : |F| \leq F\} = S_1[S_0]$. If $S_\alpha$ has been defined let $S_{\alpha+1} = S_1[S_\alpha]$. If $\alpha$ is a limit ordinal with $S_{\alpha'}$ defined for each $\alpha' < \alpha$ choose an increasing sequence of ordinals with $\alpha = \sup_n \alpha_n$ and let $S_\alpha = \cup_{n=1}^{\infty} \{F \in S_{\alpha_n} : n \leq F\}$.

For $n \geq 1$ let $(S_\alpha)^n = \{F = \bigcup_{i=1}^{n} F_i : F_i \in S_\alpha, \ F_1 < \cdots < F_n\} \text{ and let } [S_\alpha]^n = S_{\alpha}[\ldots[S_\alpha]] \ (n \text{ times})$. A sequence $(E_i)^n_1$ of finite subsets of integers is $S_\alpha$ admissible if $E_1 < \cdots < E_n$ and $(\min E_i)^n_1 \in S_\alpha$.

Note that $(S_\alpha, \subseteq)$ forms a tree, $\text{Tree}(S_\alpha)$, of order $\omega^\alpha$ and $[S_\alpha]^n$ forms a tree, $\text{Tree}([S_\alpha]^n)$, of order $\omega^{\alpha \cdot n}$ (see eg. [AA]).

**Definition 5.17.** Tsirelson spaces, $T(S_\alpha, 1/2)$ [A].

We first define $c_00$ to be the linear space of all real sequences with finite support, and let $(e_i)_{i=1}^{\infty}$ be the unit vector basis of $c_00$. If $E \subset \mathbb{N}$, then let $Ex = \sum_{i \in E} a_i e_i$.

Using the Schreier sets, Argyros defined the Tsirelson spaces, $T(S_\alpha, 1/2)$, for $\alpha < \omega_1$. He showed there exists a norm $\| \cdot \|$ on $c_00$ satisfying the implicit equation

$$\|x\| = \max \left(\|x\|_{c_00}, \frac{1}{2} \sup \left\{\sum_{i=1}^{n} \|E_i x\| : (E_i)^n_1 \text{ is } S_\alpha \text{ admissible and } n \geq 1\right\}\right).$$
The space $T(S_\alpha, 1/2)$ is the completion of $(c_{00}, \| \cdot \|)$. The standard Tsirelson space $T$ (the dual of Tsirelson’s original space $[T]$) is just $T(S_1, 1/2)$ [FJ].

**Definition 5.18.** Schreier spaces, $X_\alpha$.

The Schreier spaces are generalizations of Schreier’s example [Sch], first discussed in [AA] and [AO]. They are defined in a similar way to Tsirelson space; for each $\alpha < \omega_1$ we define a norm on $c_{00}$ by:

$$\left\| \sum a_i e_i \right\|_\alpha = \sup_{E \in S_\alpha} \left| \sum_{i \in E} a_i \right|,$$

and then the Schreier space $X_\alpha$ is the completion of $(c_{00}, \| \cdot \|_\alpha)$.

**Theorem 5.19.** $I_0(T(S_\alpha, 1/2)) = \omega^{\alpha \cdot \omega} = I(T(S_\alpha, 1/2))$.

**Proposition 5.20.** For each $\alpha < \omega_1$, for every $\varepsilon > 0$, and for all $m \geq 1$, there exists $n \geq 1$ such that if $T$ is a block basis tree on a Banach space with a basis, and if $\mathcal{F}(T) = \{(\min(\text{supp} x_i))_1^l : (x_i)_1^l \in T\}$ satisfies: $\forall F \in \mathcal{F}(T) \ \forall (a_i)_F \subset \mathbb{R}^+$ with $\sum_F a_i = 1$ there exists $G \in (S_\alpha)^m$ such that $G \subset F$ and $\sum_G a_i \geq \varepsilon$, then $o(T) \leq \omega^{\alpha \cdot n}$.

**Proof.** We prove the result by induction on $\alpha$. Let $\alpha = 0$. Pick $\varepsilon > 0$, $m \geq 1$ and choose $n$ so that $m/n < \varepsilon$. If $o(T) > \omega^{0 \cdot n} = n$, then there exists $F \in \mathcal{F}(T)$, $|F| > n$. Now, setting $a_i = 1/|F|$ for $i \in F$ gives $\sum_F a_i = 1$ but if $G \in (S_0)^m$, then $|G| = m$ and $\sum_G a_i = m/|F| < m/n < \varepsilon$, a contradiction.

Suppose the result is true for $\alpha$. To prove the case $\alpha + 1$ first let $\varepsilon > 0$ be arbitrary and fix $m = 1$. Let $n > 2/\varepsilon$, and let $T$ be a tree with $o(T) \geq \omega^{\alpha+1 \cdot n}$. We may assume by Lemma 3.7 that $T \cong T(n, \omega_{\alpha+1})$ and let $F : T \to T_n = \{a_1 < \cdots < a_n\}$ be the map $F^{-1}(a_1) = T_1(a_1)$ and $F^{-1}(a_i) = \bigcup_{n=1}^\infty T_n(a_i)$ $(i > 1)$ where $T_n(a_i) \cong T_{\omega_{\alpha+1}}$ and the $T_n(a_i)$’s are mutually incomparable. Fix $m_1 = 1$ and $\varepsilon_1 < 1/n$, $T_1(a_1)$ has index $\omega^{\alpha+1} > \omega^{\alpha \cdot k} \forall k$ so $\exists F_1 \in \mathcal{F}(T_1(a_1)) \exists (a_i)_F \subset \mathbb{R}^+$ such that $\sum_F a_i = 1$ and $\sum_G a_i < \varepsilon$ if $G \in (S_\alpha)^m$. Let $(x_i)_1^l \in T_1(a_1)$ be a node such that $F_1 = (\min(\text{supp} x_i))_1^l$. Then there exists $i_2$ such that $T_{i_2}(a_2) > (x_i)_1^l$. 

Choose $m_2 = \max(F_1)$ and $\varepsilon_2 < 1/n$. Repeating the process for the restricted tree $R(T_{i_2}(a_2))$ and $m_2$, $\varepsilon_2$ up to $R(T_{i_n}(a_n))$ and $m_n$, $\varepsilon_n$ we obtain $F_1 < \cdots < F_n$, $(a_j)_{F_i} \subset \mathbb{R}^+$ such that $\sum_{F_i} a_j = 1$ and $\sum_G a_j < \varepsilon_i$ if $G \subseteq F_i$ and $G \in (S_\alpha)^m$. Set $F = \bigcup_i^\infty F_i$ and $\pi_j = \frac{1}{n}a_j$ for $j \in F$. Let $G \subseteq F$, $G \in S_{\alpha+1}$. Then $G = \bigcup_i G_j$ where $r \leq G_1 < \cdots < G_r$ and $G_j \in S_\alpha$. Let $i$ be least such that $G \cap F_i \neq \emptyset$ then $r \leq \max(F_i) = m_{i+1} \leq m_l \forall l > i$. Hence if $l > i$, then $G \in (S_\alpha)^m$ and so $\sum_{G \cap F_j} \pi_j = \frac{1}{n} \sum_{G \cap F_j} a_j < \varepsilon_i/n$; further $\sum_{G \cap F_j} \pi_j \leq \sum_{F_i} \pi_j = 1/n$. Thus
\[
\sum_G \pi_j < \frac{1}{n}(1 + \varepsilon_i + \cdots + \varepsilon_n) < \frac{2}{n} < \varepsilon
\]
as we had to show.

For general $m > 1$ we use the same construction, taking $n > 2m/\varepsilon$. Then, for $G \in (S_{\alpha+1})^m$ each set in $S_{\alpha+1}$ can contribute at most $(1 + \sum \varepsilon_i)/n$ and hence we get the desired contradiction.

Let $\alpha$ be a limit ordinal and suppose the result is true for each $\alpha' < \alpha$. Let $(a_i)$ be the increasing sequence of ordinals, with $\sup_i a_i = \alpha$, which defines $S_\alpha$. Let $\varepsilon > 0$, $m = 1$, and choose $n > 2/\varepsilon$.

Suppose $\omega(T) \geq \omega^\alpha \cdot n$, and so assume $T \simeq T(n, \omega^\alpha)$; let $F : T \rightarrow T_n \equiv \{a_1 < \cdots < a_n\}$ be as before, but now with $T_n(a_i) \simeq T_{\omega^\alpha}$. From $\mathcal{F}(T_1(a_1))$ select $F_1$, $(a_i)_{F_1} \subset \mathbb{R}^+$ arbitrarily. Let $(x_i)_1 \in T_1(a_1)$ be a node such that $F_1 = (\min(\text{supp } x_i))_1$, then there exists $i \geq 1$ such that $T_i(a_2) \succ (x_i)_1$; set $t_2 = T_i(a_2)$.

Now, the result is true for each $\alpha' < \alpha$, and $o(R(t_2)) > \omega^{\alpha'} \cdot k$ for each $\alpha' < \alpha$ and every $k$, so there exists $F_2 \in \mathcal{F}(R(t_2))$, $m_2 > \max F_1$, and $(a_j)_{F_2}$ such that $\sum_{F_2} a_j = 1$ and every subset $G$ of $F_2$ which is also in $S_{\alpha m_2}$ satisfies $\sum_G a_j < \varepsilon_2$, where $\varepsilon_2$ was chosen to be less than $1/n$. Now, by [OTW], there exists $m$ such that if $G \geq m$ and $G \in S_{\alpha_i}$ for any $i < m_2$, then $G \in S_{\alpha m_2}$. Also, since $\omega^\alpha$ is a limit ordinal, we may remove a finite number of the smallest nodes of $R(t_2)$ without changing the order of the tree and so we may choose $F_2 \geq m$.

We continue in this fashion, as before, to obtain $F_1 < \cdots < F_n$, $(a_j)_{F_i}$ such that if $i \leq \max F_{i-1}$, $G \in S_{\alpha_i}$, $G \subseteq F_i$, then $\sum_G a_j < \varepsilon_i < 1/n$. Set $F = \bigcup F_i$ and $\overline{a}_j = \frac{1}{n}a_j$ for
Let $G \in S_\alpha$, then there exists $j \geq 1$ such that $G \in S_{\alpha_j}$ and $j \leq G$. As before let $i$ be least such that $G \cap F_i \neq \emptyset$, then $j < m_i$ ($l > i$) and so $\sum_{G \cap F_i} \pi_j = \frac{1}{n} \sum_{G \cap F_i} a_j < \epsilon / n$ and $\sum_{G \cap F_i} \pi_j \leq \sum_{F_i} \pi_j = 1/n$. Thus

$$\sum_{G} \pi_j < \frac{1}{n}(1 + \epsilon_1 + \cdots + \epsilon_n) < \frac{2}{n} < \epsilon$$

giving the required contradiction.

The case for $m > 1$ proceeds along similar lines as for the successor case; we just need to pick $n$ so that $m/n < \epsilon/2$. This completes the proof of the proposition.

$\Box$

**Proof of Theorem 5.19.** We first note that for each $n \geq 1$, if $E \in [S_\alpha]^n$, then $\| \sum_{i \in E} a_i e_i \| \geq 2^{-n} \sum |a_i|$, from the definition of the norm on $T(S_\alpha, 1/2)$, thus we may construct a block basis tree isomorphic to $\text{Tree}([S_\alpha]^n)$. As we noted in Definition 5.16 $o(\text{Tree}([S_\alpha]^n)) = \omega^{\alpha \cdot n}$, and hence $I_b(T(S_\alpha, 1/2)) \geq \omega^{\alpha \cdot n}$ for each $n \geq 1$, and so $I_b(T(S_\alpha, 1/2)) \geq \omega^{\alpha \cdot \omega}$.

Now, suppose $I_b(T(S_\alpha, 1/2)) > \omega^{\alpha \cdot \omega}$, then there exists an $\ell_1$-$K$-block basis tree $T$ of order $\omega^{\alpha \cdot \omega}$ and by Fact 5.3 (v) we may write $\omega^{\alpha \cdot \omega} = \omega^{\theta + 1}$ for some $\theta < \omega_1$. This is one of the fixed points of our construction by Remark 4.4 (ii). Thus for every $\epsilon > 0$ there exists an $\ell_1$-block subtree of $T$ with constant $1 + \epsilon$ and order $\omega^{\alpha \cdot \omega}$, so we may assume $T$ has constant $1 + \epsilon$ where $\epsilon < 1/10$.

Let $m = 1$ and choose $n$ from Proposition 5.20. Since $o(T) > \omega^a \cdot n$ there exist $F \in F(T)$, $F = \{n_1, \ldots, n_l\} = (\text{min supp } x_i)_1^l$ for some $(x_i)_1^l \in T$ and $(a_j)_F \subset \mathbb{R}^+$ such that $\sum_F a_j = 1$ and $\sum_G a_j < \epsilon / 3$ for each subset $G \subseteq F$ which is also in $S_\alpha$; set $x = \sum_{i=1}^l a_n x_i$. To calculate the norm of $x$ let $(E_i)_1^k$ be $S_\alpha$ admissible. Let $I = \{i : \text{supp}(x_i) \subseteq E_j \text{ for some } j\}$, let $J = \{i \leq l : i \notin I \text{ and supp}(x_i) \cap E_j \neq \emptyset \text{ for some } j\}$ and note that since $(E_i)_1^k$ is $S_\alpha$ admissible, there exist
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\[ A, B, C \in \mathcal{S}_\alpha \text{ such that } \{n_j : j \in J\} = A \cup B \cup C. \]

Now

\[
\frac{1}{2} k \sum_{j=1}^{k} \|E_j x\| \leq \frac{1}{2} \sum_{i=1}^{l} a_{n_i} \sum_{j=1}^{k} \|E_j x_i\| = \frac{1}{2} \left( \sum_{i \in I} a_{n_i} \sum_{j=1}^{k} \|E_j x_i\| + \sum_{i \in J} a_{n_i} \sum_{j=1}^{k} \|E_j x_i\| \right)
\]

\[
\leq \frac{1}{2} \sum_{i \in I} a_{n_i} + \sum_{i \in J} a_{n_i} \frac{1}{2} \sum_{j=1}^{k} \|E_j x_i\| 
\]

\[
\leq \frac{1}{2} \sum_{i \in J} a_{n_i} \|x_i\| 
\]

\[
\leq \frac{1}{2} + \sum_{j \in A \cup B \cup C} a_j 
\]

\[
\leq \frac{1}{2} + 3 \varepsilon \frac{3}{2}
\]

and hence \(\|x\| \leq 1/2 + \varepsilon\). However \((x_i)_i^1 \in T\), an \(\ell_1\)-(1 + \varepsilon)-tree and so \(\|x\| \geq 1/(1 + \varepsilon)\) a contradiction. Thus \(I_b(T(S_\alpha, 1/2)) = \omega^\omega\). \(\square\)

**Remark 5.21.** The authors have recently calculated the index of two other classes of Banach spaces. In \([JO]\) it is shown that the index for \(C(K)\), where \(K\) is a countable compact metric space, is given by

\[
I(C(\omega^\alpha)) = \begin{cases} 
\omega^{\alpha+2} & (0 \leq \alpha < \omega) \\
\omega^{\alpha+1} & (\omega \leq \alpha)
\end{cases}
\]

and for \(1 \leq \alpha < \omega_1\), and \(X_\alpha\) the Schreier space for \(\alpha\) (Definition 5.18), then \(I(X_\alpha) = \omega^{\alpha+1}\).

### 6. Final remarks

As we noted in the introduction, Theorem 1.1 is false for \(1 < p < \infty\). This is a consequence of \(\ell_p\) being arbitrarily distortable [OS]. In particular the following is true.

**Theorem 6.1.** For each \(p\), \(1 < p < \infty\), and every \(L \geq 1\), there exist \(K > 1\) and \(\alpha < \omega_1\) such that for any \(\beta < \omega_1\) there exists a Banach space \(X\) which contains an \(\ell_p\)-\(K\)-tree on \(X\) of order at least \(\beta\), but no \(\ell_p\)-\(L\)-tree of order \(\alpha\).
Proof. Fix \( L \geq 1 \); then since \( \ell_p \) is arbitrarily distortable there exists a Banach space \( X \) isomorphic to \( \ell_p \) satisfying \( d(Y, \ell_p) > 2L \) for every subspace \( Y \) of \( X \). Clearly, as \( X \) is isomorphic to \( \ell_p \), there exists some constant \( K \) so that \( X \) contains an \( \ell_p \)-K-tree on \( X \) of order \( \beta \) for each \( \beta < \omega_1 \). If the theorem is false, then for each \( \alpha < \omega_1 \) there would exist an \( \ell_p \)-tree on \( X \) of order at least \( \alpha \). This in turn would imply \([B]\) that \( X \) contains a subspace \( Y \) with \( d(Y, \ell_p) \leq L \), contradicting our original assumption. This completes the proof. \( \square \)

The finite version of Theorem 1.1 for \( \ell_p \) is true, as we mentioned in the introduction. From this and our construction of \( T_\omega \) (Definition 3.1) it is easy to see that if we have an \( \ell_p \)-K-tree \( T \) of order \( \omega \) on a Banach space \( X \), then there exists a block subtree of \( T \) which is an \( \ell_p \)-\((1 + \varepsilon)\)-tree of order \( \omega \). Thus it seems reasonable to ask the following question.

**Question 2.** For which ordinals \( \alpha \) is Theorem 1.1 true for \( 1 < p < \infty \), and what is their supremum?

**Definition 6.2.** \( \ell_p \)-\( \alpha \)-spreading models (\( S_\alpha \)-SMs)

We extend the definition of the \( \ell_1 \)-spreading models introduced by Kiriakouli and Negrepontis [KN] to \( \ell_p \) \((1 \leq p \leq \infty)\). A sequence \( (x_n)_{n=1}^\infty \) has an \( \ell_p \)-\( \alpha \)-spreading model, for some \( 1 \leq p \leq \infty \), with constant \( K \), if \( (x_i)_{i \in F} \sim uvb \ell_p^{|F|} \) for every \( F \in S_\alpha \), where \( S_\alpha \) is the collection of Schreier sets of order \( \alpha \) introduced in Section 5.

We can refine the constant of an \( \ell_1 \)-SM from \( K \) to \((1 + \varepsilon)\) on a block basis as we did above for \( \ell_1 \)-trees, but the proof is much more straightforward. We also note that these spreading models are a stronger notion than \( \ell_1 \)-trees.

We need the following result [OTW]:

**Lemma 6.3.** (OTW) For each pair \( \alpha, \beta < \omega_1 \) there exists \( N \subseteq \mathbb{N} \) such that \( S_\alpha[S_\beta](N) \subseteq S_{\beta + \alpha} \).

**Theorem 6.4.** For any \( K > 1 \), every \( \varepsilon > 0 \), and each \( \alpha < \omega_1 \), there exists \( \beta < \omega_1 \) such that if \((x_n)\) is a normalized basic sequence having an \( \ell_1 \)-\( S_\beta \)-SM with constant \( K \), then there exists a normalized block basis \( (y_n) \) of \((x_n)\) having an \( \ell_1 \)-\( S_\alpha \)-SM with constant \( 1 + \varepsilon \).
Proof. This follows immediately from the following lemma. □

**Lemma 6.5.** Let \((x_n)\) be a normalized basic sequence having an \(\ell_1\)-\(S_{\alpha,2}\)-SM with constant \(K\). Then there exists a normalized block basis \((y_n)\) of \((x_n)\) having an \(\ell_1\)-\(S_{\alpha}\)-SM with constant \(\sqrt{K}\).

**Proof.** For fixed \(\alpha < \omega_1\) choose, by Lemma 6.3, \(N = (n_i) \subseteq \mathbb{N}\) such that \(S_\alpha[S_\alpha](N) \subseteq S_{\alpha,2}\) and consider the subsequence \((x_{n_i})_1^{\infty}\). We know that since \(S_{\alpha} \cdot 2 \subseteq S_{\alpha}\),
\[
\left\| \sum_{i \in F} a_i x_{n_i} \right\| \geq \frac{1}{K} \sum_{i \in F} |a_i|, \text{ for every } (a_i) \subset \mathbb{R}, \text{ and } F \in S_{\alpha,2}.
\]

If there exists \(k \geq 1\) such that
\[
\left\| \sum_{i \in E} a_i x_{n_i} \right\| \geq \frac{1}{\sqrt{K}} \sum_{i \in E} |a_i|, \text{ for every } (a_i) \subset \mathbb{R}, \text{ and each } E \in S_\alpha \text{ with } E > k
\]
then we are finished since \(E \in S_\alpha\) implies \(E + k \in S_\alpha\) \((k \geq 1)\).

Otherwise there exists a normalized block basis \((y_j)\) of \((x_{n_i})\) satisfying
\[
y_j = \sum_{i \in E_j} a_i x_{n_i}, \sum_{i \in E_j} |a_i| > \sqrt{K}
\]
with \(E_j \in S_\alpha\) and \(E_j < E_{j+1}\) for each \(j \geq 1\). Now, for each \(E \in S_\alpha\) the set \(F = \bigcup_{j \in E} E_j\) is an element of \(S_\alpha[S_\alpha](N)\), which in turn is contained in \(S_{\alpha,2}\). Thus we obtain
\[
\left\| \sum_E b_j y_j \right\| \geq \frac{1}{\sqrt{K}} \sum_E |b_j|, \text{ for every } (b_j) \subset \mathbb{R}, \text{ and } E \in S_\alpha
\]
using James’ argument as in the proof of Theorem 1.1. □

**Remark 6.6.** We note here some closing points for this section.

(i) For every \(\alpha < \omega_1\) there exists a Banach space \(X_\alpha\) with an \(\ell_1\)-tree of order \(\alpha\) but \(X_\alpha\) has no \(\ell_1\)-spreading models. In fact \(X_\alpha\) can be taken to be reflexive with all normalized weakly null sequences having an \(\ell_2\)-(1 + \(\varepsilon\)) subsequence.
Proof. We use a similar construction to Szlenk [Sz]. Let \( X_k = \ell^k_1 \) (\( k \geq 1 \)). If \( \alpha < \omega_1 \) is a limit ordinal and we have constructed \( X_\beta \) for each \( \beta < \alpha \) let \( X_\alpha = (\sum_{\beta < \alpha} X_\beta)_{\ell^2_1} \). Given \( X_\alpha \) let \( X_{\alpha+1} = \left( X_\alpha \oplus \mathbb{R} \right)_{\ell^1_1} \).

(ii) As for the \( \ell^p \)-trees, Theorem 6.4 is also true for \( p = \infty \) and false for \( 1 < p < \infty \). This follows from the proof of Theorem 6.1.

(iii) It follows from Lemma 6.5 that if \((x_n)\) is a normalized basic sequence having an \( \ell^1 \)-SM with any constant, then for every \( \beta < \omega_\alpha \), and any \( \varepsilon > 0 \), there exists a normalized block basis \((y_n)\) of \((x_n)\) having an \( \ell^1 \)-SM with constant \( 1 + \varepsilon \).

References

[A] ARGYROS, S., Banach spaces of the type of Tsirelson.


[KN] KIRIAKOULI, P. & NERGREPONTIS, S., Baire-1 functions and spreading models of \( \ell^1 \). Preprint.


[OTW] ODELL, E. & TOMCZAK-JAEGERMANN, N. & WAGNER, R., Proximity to $\ell_1$ and distortion in asymptotic $\ell_1$ spaces. *Preprint*.


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